

STABILITY ANALYSIS AND BEST APPROXIMATION ERROR ESTIMATES OF DISCONTINUOUS TIME-STEPPING SCHEMES FOR THE ALLEN-CAHN EQUATION

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ABSTRACT. Fully-discrete approximations of the Allen-Cahn equation are considered. In particular, we consider schemes of arbitrary order based on a discontinuous Galerkin (in time) approach combined with standard conforming finite elements (in space). We prove best approximation a-priori error estimates, with constants depending polynomially upon $(1/\epsilon)$ by circumventing Grönwall Lemma arguments. We also prove that these schemes are unconditionally stable under minimal regularity assumptions on the given data. The key feature of our approach is an appropriate duality argument, combined with a boot-strap technique.

1. INTRODUCTION

The Allen-Cahn equation is a parameter dependent parabolic semi-linear PDE of the form

$$(1.1) \quad \begin{cases} u_t - \Delta u + \frac{1}{\epsilon^2}(u^3 - u) &= f & \text{in } (0, T) \times \Omega \\ u &= 0 & \text{on } (0, T) \times \Gamma \\ u(0, x) &= u_0 & \text{in } \Omega. \end{cases}$$

Here, Ω denotes a bounded domain in \mathbb{R}^d , $d = 2, 3$ with Lipschitz boundary Γ , u_0 and f denote the initial data and the forcing term respectively. The principal difficulty involved, concerns the parameter $0 < \epsilon \ll 1$ appearing in the model problem, which is typically very small and comparable to the size of the time and space discretization parameters, τ, h respectively. The Allen-Cahn equation can be viewed as the simplest phase field model, and was introduced in [2].

Our main goal is to provide a rigorous stability analysis of a general class of fully-discrete schemes and to prove best approximation a-priori error estimates. The schemes considered here are discontinuous (in time) and conforming in space. The motivation for using the discontinuous (in time) Galerkin approach relies in its robust performance in a vast area of problems whose solutions satisfy low regularity properties.

The key feature of the discontinuous time stepping Galerkin schemes is their ability to mimic the stability properties of the corresponding continuous system. Indeed, we prove that the fully-discrete solution, computed by using discontinuous Galerkin (in time) and conforming finite elements in space of arbitrary order (in

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time and space), (denoted by u_h) satisfies the following unconditional stability estimates:

$$\|u_h\|_{L^2[0,T;L^2(\Omega)]} \leq C, \quad \text{and} \quad \|u_h\|_{L^\infty[0,T;L^2(\Omega)]} + \|u_h\|_{L^2[0,T;H^1(\Omega)]} \leq \frac{C}{\epsilon},$$

where C denotes a constant depending on the domain Ω , the norms of $\|u_0\|_{L^2(\Omega)}$ and $\|f\|_{L^2[0,T;H^{-1}(\Omega)]}$ and the polynomial degree in time, but it is independent of τ, h, ϵ .

In addition, using the above estimates, we are able to prove the following best-approximation error estimate,

$$\|\text{error}\|_X \leq \frac{C}{\epsilon} (\|u\|_{L^\infty[0,T;H^1(\Omega)]}^2 + \|u\|_{L^2[0,T;H^2(\Omega)]}^2) \|\text{best approximation error}\|_X,$$

where $X = L^\infty[0,T;L^2(\Omega)] \cap L^2[0,T;H^1(\Omega)]$, and C denotes an algebraic constant depending only upon data, and it is independent of τ, h, ϵ . For the above best approximation error estimate the spacial and temporal discretization parameters (denoted by h and τ) satisfy $\tau^{1/2} + h \leq C\epsilon^2$ when $d = 2$, and $\tau^{1/2} + h \leq C\epsilon^{8/3}$ when $d = 3$.

The above estimate states that the error is as good as the approximation properties of the underlying subspaces, and the regularity of the solution will allow it to be. Therefor, it can be viewed as a generalization of the the classical Cea's Lemma. In addition, we also prove improved estimates in the $L^2[0,T;L^2(\Omega)]$ norm.

The Allen-Cahn equation represents a model problem which possess structural difficulties which significantly complicate the numerical analysis of any potential scheme. In particular, the physical phenomena modeled by the above system, posses complex dynamics for realistic values of the parameter $\epsilon \ll 1$. For instance, we note that the natural energy norms $\|\cdot\|_{L^\infty[0,T;L^2(\Omega)]}$, $\|\cdot\|_{L^2[0,T;H_0^1(\Omega)]}$ imposed by the structure of our problem scale differently in terms of the parameter ϵ , compared to the norm of $\|\cdot\|_{L^4[0,T;L^4(\Omega)]}$ that naturally arises from the nonlinear term. In addition, the presence of $L^2[0,T;L^2(\Omega)]$ norm with the “wrong sign” poses a substantial difficulty in the analysis as well as in the numerical analysis of fully-discrete schemes for such problem: Classical techniques based on Gronwall's type of inequalities typically fail, since they introduce constants depending on quantities of $\exp(1/\epsilon)$. This problem was first circumvented in the works of [3, 6, 12] by developing uniform bounds of the principal eigenvalue of the linearized Allen-Cahn operator, i.e., bounds for the quantity $\inf_{0 \neq v \in H^1(\Omega)} \frac{\|\nabla v\|_{L^2(\Omega)}^2 + ((3u^2(t)-1)v, v)}{\|v\|_{L^2(\Omega)}^2}$

which are available when the Allen-Cahn equation describes a smooth evolution of a developing interface.

Based on the above idea, for the numerical analysis of the implicit Euler scheme, in [19], were established the first a-priori bounds in various norms with constants that depend upon $(1/\epsilon)$ in a polynomial fashion. For instance, for the energy norm an estimate of order $\tau + h$ with constant depending upon $1/\epsilon^3$, when the data $\|\nabla u_0\|_{L^2(\Omega)}$, $\|\Delta u_0\|_{L^2(\Omega)}$, $\lim_{s \rightarrow 0^+} \|\nabla u_t(s)\|_{L^2(\Omega)} \leq C$ and the spacial and the temporal discretization parameters satisfy $\tau + h^2 \leq C\epsilon^7$ and $h|lnh|^{1/2} \leq \epsilon^3$ when $d = 2$ and $\tau + h^2 \leq \epsilon^{13}$, and $h \leq \epsilon^6$, when $d=3$ respectively. The idea of using the principal eigenvalue operator, was further used in order to obtain a-posteriori bounds in [25], and [20], while various a-posteriori estimates based on a discretized version of the principal eigenvalue operator where obtained in the works of [4], [5], and [21]. In [28], semi-implicit schemes of first order were studied, and conditional stability

estimates were presented for semi-discrete (in time) approximations. In addition, a second order semi-implicit, semi-discrete in time scheme which is conditionally stable was also considered, in [28]. Finally, extensive numerical studies of various numerical schemes for the Allen-Cahn equation are presented in [24, 33]. For various results regarding discontinuous time-stepping schemes for the semi-linear parabolic pdes, we refer the reader to [15, 16, 17, 30].

To our best knowledge, so far, in the literature, there has been no rigorous proof regarding unconditional stability as well as best-approximation type of error estimates for any kind of fully-discrete scheme with polynomial dependence on the quantity $(1/\epsilon)$. The scope of this work, is to prove that for a very broad category of fully-discrete schemes such (unconditional) stability estimates as well as best approximation error estimates (in the spirit of the classical Cea's Lemma) are possible, even under low regularity assumptions on the given data.

We close our introduction, by introducing the main idea which is essential for the analysis of our estimates. For the stability analysis, instead of focusing on the uniform bounds of the principal eigenvalue of the linearized (elliptic) part of the Allen-Cahn operator, we construct a fully-discrete linearized Allen-Cahn equation with an appropriately scaled $L^2[0, T; L^2(\Omega)]$ part, based on the discontinuous time-stepping Galerkin formulation. This auxiliary space-time projection effectively allows to apply a duality argument, to recover first the unconditional stability with respect to $L^2[0, T; L^2(\Omega)]$ norm, and then a boot-strap argument to recover the unconditional stability in $L^2[0, T; H^1(\Omega)]$ and $L^\infty[0, T; L^2(\Omega)]$. For the later we employ the techniques developed by [8, 9, 31], in a way to avoid the use of a Grönwall's type argument. The discrete compactness argument of Walkington (see [31]), then allows to rigorously pass the limit to prove convergence. We note that the case of zero Neumann boundary data can be also considered in an identical way. For the best approximation error estimate we employ a similar strategy. First, we define an auxiliary space-time (linear) parabolic projection that exhibits best approximation error estimates, and then we use a duality argument, and the stability estimates in crucial way, in order to obtain a preliminary (suboptimal) estimate for the $L^2[0, T; L^2(\Omega)]$ norm without using Grönwall's type of arguments. Then, we recover the full rate in the $L^2[0, T; H^1(\Omega)]$ norm via a boot-strap argument and the estimate at arbitrary time-points via the techniques developed by [8, 9, 31] to obtain the symmetric structure of the best-approximation error estimate. Finally, we recover the improved rate for the $L^2[0, T; L^2(\Omega)]$ norm by completing the boot-strap argument.

2. PRELIMINARIES

2.1. Notation. Let U denote a Banach space. Typically, $U \equiv H^s(\Omega)$, $0 < s \in \mathbb{R}$, where $H^s(\Omega)$ denotes the standard Sobolev (Hilbert) spaces (see for instance [18, 32]). We denote by $H^0(\Omega) \equiv L^2(\Omega)$, and by $H_0^1(\Omega) \equiv \{w \in H^1(\Omega) : w|_\Gamma = 0\}$. Finally, we use the notation $\langle \cdot, \cdot \rangle$ for the duality pairing of $H^{-1}(\Omega)$, $H_0^1(\Omega)$ and (\cdot, \cdot) for the standard L^2 inner product, where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. We denote the time-space spaces by $L^p[0, T; U]$, $L^\infty[0, T; U]$, endowed with norms:

$$\|w\|_{L^p[0, T; U]} = \left(\int_0^T \|w\|_U^p dt \right)^{\frac{1}{p}}, \quad \|w\|_{L^\infty[0, T; U]} = \text{esssup}_{t \in [0, T]} \|w\|_U.$$

The set of all continuous functions $v : [0, T] \rightarrow U$, is denoted by $C[0, T; U]$ with norm $\|w\|_{C[0, T; U]} = \max_{t \in [0, T]} \|w(t)\|_U$. For the definition of spaces $H^s[0, T; U]$, we refer the reader to [18, 32]. Throughout this work we will use the following (natural energy) space for the solution u of (1.1),

$$X = L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H_0^1(\Omega)]$$

with associated norm $\|w\|_X^2 = \|w\|_{L^\infty[0, T; L^2(\Omega)]}^2 + \|w\|_{L^2[0, T; H_0^1(\Omega)]}^2$. The bilinear form related to our problem is defined by

$$a(w_1, w_2) = \int_{\Omega} \nabla w_1 \nabla w_2 dx \quad \forall w_1, w_2 \in H_0^1(\Omega).$$

Using Poincaré's inequality we obtain the corresponding coercivity condition

$$a(w, w) \geq C \|w\|_{H_0^1(\Omega)}^2 \quad \forall w \in H_0^1(\Omega),$$

where C denotes an algebraic constant depending only upon the domain Ω . We close this preliminary section, by recalling Young's inequality and Landyzyeskaya-Gagliardo-Nirenberg interpolation inequalities.

Young's Inequality: For any $a, b \geq 0$ any $\delta > 0$, and $s_1, s_2 > 1$

$$ab \leq \delta a^{s_1} + C(s_1, s_2) \delta^{-\frac{s_2}{s_1-1}} b^{s_2}, \quad \text{where } (1/s_1) + (1/s_2) = 1.$$

Landyzyeshkayka-Gagliardo-Nirenberg Interpolation Inequalities: There exist constant $C > 0$ depending only upon the domain such that, for all $u \in H_0^1(\Omega)$,

$$\begin{aligned} \|u\|_{L^4(\Omega)} &\leq C \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2}, & \text{when } d = 2, \\ \|u\|_{L^3(\Omega)} &\leq C \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2}, & \text{when } d = 3, \\ \|u\|_{L^4(\Omega)} &\leq C \|u\|_{L^2(\Omega)}^{1/4} \|u\|_{H^1(\Omega)}^{3/4}, & \text{when } d = 3. \end{aligned}$$

2.2. Weak formulation and regularity of the Allen-Cahn equation. The following weak formulation of (1.1) will be used subsequently. Let $f \in L^2[0, T; H^{-1}(\Omega)]$ and $u_0 \in L^2(\Omega)$. Then, for all $w \in H_0^1(\Omega)$ and for a.e. $t \in (0, T]$, we seek $u \in L^2[0, T; H_0^1(\Omega)] \cap H^1[0, T; H^{-1}(\Omega)]$ such that

$$\langle u_t, w \rangle + a(u, w) + (1/\epsilon^2) \langle u^3 - u, w \rangle = \langle f, w \rangle, \quad \text{and} \quad (u(0), w) = (u_0, w).$$

Since, our schemes are based on the discontinuous time-stepping framework, a suitable weak formulation can be written as follows: We seek $u \in L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H_0^1(\Omega)]$, satisfying,

$$\begin{aligned} (2.1) \quad & (u(T), w(T)) + \int_0^T \left(-\langle u, w_t \rangle + a(u, w) + \frac{1}{\epsilon^2} \langle u^3 - u, w \rangle \right) dt \\ & = (u_0, w(0)) + \int_0^T \langle f, w \rangle dt \end{aligned}$$

for all $w \in L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$. It clear that using straightforward techniques (see for instance [29, 32]) one can easily prove the existence a weak solution $u \in L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H_0^1(\Omega)]$ which satisfies the following energy estimate

$$\|u\|_X \leq C_\epsilon \left(\|f\|_{L^2[0, T; H^{-1}(\Omega)]} + \|u_0\|_{L^2(\Omega)} \right),$$

where C_ϵ depends on Ω , and the parameters ϵ and T .

The following Lemma quantifies the dependence upon ϵ of various norms.

Lemma 2.1. Suppose that $f \in L^2[0, T; H^{-1}(\Omega)]$ and $u_0 \in L^2(\Omega)$. Then, there exists a constant C (independent of ϵ) such that:

$$\begin{aligned} \|u\|_{L^2[0, T; L^2(\Omega)]} + \|u\|_{L^4[0, T; L^4(\Omega)]}^2 &\leq C \left(T^{1/2} + \epsilon(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2[0, T; H^{-1}(\Omega)]}) \right), \\ \|u\|_{L^\infty[0, T; L^2(\Omega)]} + \|u\|_{L^2[0, T; H^1(\Omega)]} &\leq \frac{C}{\epsilon}. \end{aligned}$$

Suppose that

$$(2.2) \quad f \in L^2[0, T; L^2(\Omega)] \quad \text{and} \quad \|\nabla u_0\|_{L^2(\Omega)} + \frac{1}{\epsilon^2} \|(1/4)(u_0^2 - 1)\|_{L^1(\Omega)} \leq C.$$

Then, there exists a constant C (independent of ϵ) such that the following estimate holds:

$$(2.3) \quad \|u\|_{L^2[0, T; H^2(\Omega)]} \leq \frac{C}{\epsilon}, \quad \|u\|_{L^\infty[0, T; H^1(\Omega)]} + \|u_t\|_{L^2[0, T; L^2(\Omega)]} \leq C.$$

Proof. For the first estimate, we use the following auxiliary backwards in time linear parabolic pde. Let u be the solution of (2.1). Given, right hand side $u \in L^2[0, T; L^2(\Omega)]$ and terminal data $\phi(T) = 0$, we seek $\phi \in L^2[0, T; H_0^1(\Omega)] \cap H^1[0, T; H^{-1}(\Omega)]$ such that, for all $w \in L^2[0, T; H_0^1(\Omega)] \cap H^1[0, T; H^{-1}(\Omega)]$,

$$(2.4) \quad \int_0^T \left((\phi, w_t) + a(\phi, w) + \frac{1}{\epsilon^2} (u^2 \phi, w) + \frac{1}{\epsilon^2} (\phi, w) \right) dt + (\phi(0), w(0)) = \int_0^T (u, w) dt.$$

It is clear that setting $w = \phi$ in (2.4) we obtain the following bound:

$$\begin{aligned} &\frac{1}{2} \|\phi(0)\|_{L^2(\Omega)} + C \|\phi\|_{L^2[0, T; H^1(\Omega)]} + \frac{1}{\epsilon} \|\phi u\|_{L^2[0, T; L^2(\Omega)]} + \frac{1}{2\epsilon} \|\phi\|_{L^2[0, T; L^2(\Omega)]} \\ (2.5) \quad &\leq \frac{\epsilon}{2} \|u\|_{L^2[0, T; L^2(\Omega)]}. \end{aligned}$$

Now, we employ a “duality” argument. Integrating by parts in time (2.1), and setting $w = \phi$ into the resulting equation, we obtain:

$$(2.6) \quad \int_0^T \left(\langle u_t, \phi \rangle + a(u, \phi) + \frac{1}{\epsilon^2} (u^3 - u, \phi) \right) dt = \int_0^T \langle f, \phi \rangle dt.$$

Setting $w = u$ into (2.4) and subtracting the resulting equality from (2.6) we derive:

$$(2.7) \quad \int_0^T \|u\|_{L^2(\Omega)}^2 dt = \frac{2}{\epsilon^2} \int_0^T (\phi, u) dt + \int_0^T \langle f, \phi \rangle dt + (\phi(0), u(0)).$$

Note that using Hölder’s inequality, and the stability estimates, equation (2.7) implies that

$$\begin{aligned} \|u\|_{L^2[0, T; L^2(\Omega)]}^2 &\leq \frac{2}{\epsilon^2} \int_0^T |\Omega|^{1/2} \|\phi u\|_{L^2(\Omega)} dt \\ &\quad + \|f\|_{L^2[0, T; H^{-1}(\Omega)]} \|\phi\|_{L^2[0, T; H^1(\Omega)]} + \|\phi(0)\|_{L^2(\Omega)} \|u(0)\|_{L^2(\Omega)} \\ &\leq \frac{2}{\epsilon^2} |\Omega|^{1/2} T^{1/2} \|\phi u\|_{L^2[0, T; L^2(\Omega)]} \\ &\quad + C(\|f\|_{L^2[0, T; H^{-1}(\Omega)]} + \|u(0)\|_{L^2(\Omega)}) \epsilon \|u\|_{L^2[0, T; L^2(\Omega)]} \\ &\leq \frac{2}{\epsilon^2} |\Omega|^{1/2} T^{1/2} \frac{\epsilon^2}{2} \|u\|_{L^2[0, T; L^2(\Omega)]}^2 \\ &\quad + C(\|f\|_{L^2[0, T; H^{-1}(\Omega)]} + \|u(0)\|_{L^2(\Omega)}) \epsilon \|u\|_{L^2[0, T; L^2(\Omega)]}, \end{aligned}$$

which implies the desired estimate on $\|u\|_{L^2[0,T;L^2(\Omega)]}$. Returning back to (2.1), setting $w = u$, and using the bound on $\|u\|_{L^2[0,T;L^2(\Omega)]}$ we obtain the first estimate. For the second estimate, we set $w = u_t$, and we observe,

$$\int_0^T \left(\|u_t\|_{L^2(\Omega)}^2 + \frac{d}{dt} \left(\frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon^2} \|(u^2 - 1)^2\|_{L^1(\Omega)} \right) \right) dt = \int_0^T (f, u_t) dt.$$

The estimate now follows by standard algebra. The estimate on $\|\Delta u\|_{L^2[0,T;L^2(\Omega)]}$ follows using standard techniques. \square

Remark 2.2. If more regularity is available, then we can quantify the dependence upon $1/\epsilon$ in other norms (see for instance [19, Proposition 1]). In addition to (2.2), if the initial data satisfy, $\|\Delta u_0\|_{L^2(\Omega)} \leq C$ with constant C independent of ϵ , then,

$$\|u\|_{L^\infty[0,T;H^2(\Omega)]} + \|u_t\|_{L^\infty[0,T;L^2(\Omega)]} \leq \frac{C}{\epsilon}, \quad \|\nabla u_t\|_{L^2[0,T;L^2(\Omega)]} \leq \frac{C}{\epsilon}.$$

We point out that the regularity bound on $\frac{1}{\epsilon^2} \|(1/4)(u_0^2 - 1)^2\|_{L^1(\Omega)} \leq C$ is essential in order to obtain (2.3). It is worth noting that if only $\|u_0\|_{H^1(\Omega)} \leq C$ is assumed then the dependence upon $\frac{1}{\epsilon}$ deteriorates to:

$$\|u\|_{L^\infty[0,T;H^1(\Omega)]} + \|u_t\|_{L^2[0,T;L^2(\Omega)]} + \|u\|_{L^2[0,T;H^2(\Omega)]} \leq \frac{C}{\epsilon^2}.$$

For the stability analysis of the fully-discrete schemes, enhanced regularity assumptions, such as $u \in L^\infty[0,T;H^2(\Omega)] \cap H^1[0,T;H^1(\Omega)]$ are not necessary. For the error estimates, the constants will depend upon the norms of $\|u\|_{L^\infty[0,T;H^1(\Omega)]}$, $\|u_t\|_{L^2[0,T;L^2(\Omega)]}$ and $\|u\|_{L^2[0,T;H^2(\Omega)]}$.

3. THE FULLY-DISCRETE SCHEME

3.1. The discontinuous time-stepping approximations. For the discretization of the Allen-Cahn model we employ a discontinuous time-stepping Galerkin approach, combined with standard conforming finite element subspaces. Approximations will be constructed on a partition $0 = t^0 < t^1 < \dots < t^N = T$ of $[0, T]$. On each interval of the form $(t^{n-1}, t^n]$ of length $\tau_n = t^n - t^{n-1}$, a subspace U_h of $H_0^1(\Omega)$ is specified for all $n = 1, \dots, N$ and it is assumed that each U_h satisfies the classical approximation theory results (see e.g. [10]), on regular meshes. In particular, we assume that there exists an integer $\ell \geq 1$ and a constant $c > 0$ (independent of the mesh-size parameter h) such that if $w \in H^{l+1}(\Omega) \cap H_0^1(\Omega)$,

$$\inf_{w_h \in U_h} \|w - w_h\|_{H^s(\Omega)} \leq Ch^{l+1-s} \|w\|_{H^{l+1}(\Omega)}, \quad 0 \leq l \leq \ell, \quad s = -1, 0, 1.$$

We also assume that the partition is quasi-uniform in time, i.e., there exists a constant $0 < \theta \leq 1$ such that $\theta\tau \leq \min_{n=1,\dots,N} \tau_n$, where $\tau = \max_{n=1,\dots,N} \tau_n$. We seek approximate solutions which belong to the space

$$\mathcal{U}_h = \{w_h \in L^2[0, T; H_0^1(\Omega)] : w_h|_{(t^{n-1}, t^n]} \in \mathcal{P}_k[t^{n-1}, t^n; U_h]\}.$$

Here $\mathcal{P}_k[t^{n-1}, t^n; U_h]$ denotes the space of polynomials of degree k or less having values in U_h^n . By convention, the functions of \mathcal{U}_h are left continuous with right limits and hence will subsequently write w_{h-}^n for $w_h(t^n) = w_h(t_-^n)$, and w_{h+}^n for $w_h(t_+^n)$. Note that, we have also used the following notational abbreviation, $w_h \equiv w_{h,\tau}$, $\mathcal{U}_h \equiv \mathcal{U}_{h,\tau}$ etc, since for the stability analysis we will not impose any restriction involving τ , and h . The jump at t^n will be denoted as $[w_h^n] = w_{h+}^n - w_{h-}^n$. The fully

discrete system is defined as follows: We seek $u_h \in \mathcal{U}_h$ such that for every $w_h \in \mathcal{U}_h$ and for $n = 1, \dots, N$,

$$(3.1) \quad \begin{aligned} & (u_{h-}^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} \left(-\langle u_h, w_{ht} \rangle + a(u_h, w_h) + (1/\epsilon^2)(u_h^3 - u_h, w_h) \right) dt \\ & = (u_{h-}^{n-1}, w_{h+}^{n-1}) + \int_{t^{n-1}}^{t^n} \langle f, w_h \rangle dt. \end{aligned}$$

Recall that f, u_0 are given data, and u^0 denotes approximations of u_0 . In our case, we will define $u^0 = P_h u^0$, where P_h denotes the standard L^2 projection, i.e., $P_h : L^2(\Omega) \rightarrow U_h$, $(P_h v - v, w_h) = 0, \quad \forall w_h \in U_h$.

Remark 3.1. For any $\epsilon > 0$, existence and uniqueness of discontinuous Galerkin approximations of (3.1) can be proved easily (even for more complicated nonlinearities) due to finite dimensionality of the problem. For several results regarding discontinuous time-stepping schemes, with linear and semi-linear terms, we refer the reader to the works [1, 11, 13, 14, 15, 23, 27, 30, 31] (see also references within).

3.2. The basic estimate using duality. We begin by developing a stability estimate via duality for the $L^2[0, T; L^2(\Omega)]$ norm. For this purpose, we define a backwards in time parabolic problem with right hand side $u_h \in L^2[0, T; L^2(\Omega)]$ with an enhanced $L^2[0, T; L^2(\Omega)]$ term and zero terminal data. In particular, for right hand side $u_h \in L^2[0, T; L^2(\Omega)]$, and terminal data $\phi_{h+}^N = 0$, we seek $\phi_h \in \mathcal{U}_h$ such that for all $w_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h]$, and for $N = 1, \dots, 1$,

$$(3.2) \quad \begin{aligned} & -(\phi_{h+}^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} ((\phi_h, w_{ht}) + a(\phi_h, w_h) + (1/\epsilon^2)\langle u_h^2 \phi_h, w_h \rangle) \\ & + \int_{t^{n-1}}^{t^n} (1/\epsilon^2)(\phi_h, w_h) dt + (\phi_{h+}^{n-1}, w_{h+}^{n-1}) = \int_{t^{n-1}}^{t^n} (u_h, w_h) dt. \end{aligned}$$

Note that is easy to prove existence at partition points as well as in $L^2[0, T; H_0^1(\Omega)]$, due to the signs of the inner products $(1/\epsilon^2)(u_h^2 \phi_h, w_h)$ and $(1/\epsilon^2)(\phi_h, w_h)$. Given, $u_h \in \mathcal{U}_h$, it is obvious that $\phi_h \in \mathcal{U}_h$ is unique. In Section 4.2, we will also prove that $u_h \in L^\infty[0, T; L^2(\Omega)]$.

Lemma 3.2. Let $f \in L^2[0, T; H^{-1}(\Omega)]$, $u_0 \in L^2(\Omega)$, and $u_h, \phi_h \in \mathcal{U}_h$ are the solutions of (3.1)-(3.2) respectively. Then, there exists a constant $C > 0$, depending only upon the domain Ω , T , and which is independent of ϵ such that:

$$\|u_h\|_{L^2[0, T; L^2(\Omega)]} \leq C \left(T^{1/2} + \epsilon(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2[0, T; H^{-1}(\Omega)]}) \right)$$

In addition, the following estimates hold: For all $n = 1, \dots, N$

$$\begin{aligned} & \|u_{h-}^n\|_{L^2(\Omega)} + \|u_h\|_{L^2[0, T; H^1(\Omega)]} + (1/\epsilon)\|u_h\|_{L^4[0, T; L^4(\Omega)]}^2 \\ & \leq (C/\epsilon) (\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2[0, T; H^{-1}(\Omega)]}) . \end{aligned}$$

where C is a constant depending only upon Ω, T .

Proof. Setting $w_h = \phi_h$, into (3.2), using Young's inequality to bound

$$\int_{t^{n-1}}^{t^n} (u_h, v_h) dt \leq (1/2\epsilon^2) \int_{t^{n-1}}^{t^n} \|\phi_h\|_{L^2(\Omega)}^2 + (\epsilon^2/2) \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)}^2 dt,$$

and adding the resulting terms, we derive the following estimate. For all $n = N, \dots, 1$

$$(3.3) \quad \begin{aligned} & \|\phi_{h+}^{n-1}\|_{L^2(\Omega)}^2 + \|\nabla \phi_h\|_{L^2[0,T;L^2(\Omega)]}^2 + (1/\epsilon^2)\|\phi_h u_h\|_{L^2[0,T;L^2(\Omega)]}^2 \\ & + (1/2\epsilon^2)\|\phi_h\|_{L^2[0,T;L^2(\Omega)]}^2 \leq (\epsilon^2/2)\|u_h\|_{L^2[0,T;L^2(\Omega)]}^2. \end{aligned}$$

Now setting $w_h = u_h$ into (3.2), we easily derive

$$\begin{aligned} & -(\phi_{h+}^n, u_{h-}^n) + \int_{t^{n-1}}^{t^n} ((\phi_h, u_{ht}) + a(u_h, \phi_h) + (1/\epsilon^2)\langle u_h^2 \phi_h, u_h \rangle + (1/\epsilon^2)(\phi_h, u_h)) dt \\ & + (\phi_{h+}^{n-1}, u_{h+}^{n-1}) = \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)}^2 dt. \end{aligned}$$

Integrating by parts in time, we deduce,

$$(3.4) \quad \begin{aligned} & -(\phi_{h+}^n, u_{h-}^n) + (\phi_{h-}^n, u_{h-}^n) + \int_{t^{n-1}}^{t^n} (-\langle \phi_{ht}, u_h \rangle + a(\phi_h, u_h)) dt \\ & + \int_{t^{n-1}}^{t^n} ((1/\epsilon^2)\langle u_h^2 \phi_h, u_h \rangle + (1/\epsilon^2)(\phi_h, u_h)) dt = \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)}^2 dt. \end{aligned}$$

Setting $w_h = \phi_h$ into (3.1), we obtain,

$$(3.5) \quad \begin{aligned} & (u_{h-}^n, \phi_{h-}^n) + \int_{t^{n-1}}^{t^n} (-\langle u_h, \phi_{ht} \rangle + a(u_h, \phi_h) + (1/\epsilon^2)\langle u_h^3 - u_h, \phi_h \rangle) dt \\ & = (u_{h-}^{n-1}, \phi_{h+}^{n-1}) + \int_{t^{n-1}}^{t^n} \langle f, \phi_h \rangle dt. \end{aligned}$$

Subtracting (3.5) from (3.4), and noting that the terms $(1/\epsilon^2) \int_{t^{n-1}}^{t^n} \int_{\Omega} u_h^3 \phi_h dx dt$ are canceled, we arrive to

$$(3.6) \quad \begin{aligned} & (\phi_{h+}^n, u_{h-}^n) - (u_{h-}^{n-1}, \phi_{h+}^{n-1}) + \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)}^2 dt \\ & = (1/\epsilon^2) \int_{t^{n-1}}^{t^n} (\phi_h, u_h) dt - \int_{t^{n-1}}^{t^n} \langle f, \phi_h \rangle dt + (1/\epsilon^2) \int_{t^{n-1}}^{t^n} (u_h, \phi_h) dt. \end{aligned}$$

First, we treat the terms involving $(1/\epsilon^2)$ constants. Using Young's inequality with appropriate $\delta_1 > 0$ (to be determined later), we deduce,

$$\begin{aligned} & (2/\epsilon^2) \int_{t^{n-1}}^{t^n} |(\phi_h, u_h)| dt \leq (2/\epsilon^2) \int_{t^{n-1}}^{t^n} |\Omega|^{1/2} \|\phi_h u_h\|_{L^2(\Omega)} dt \\ & \leq (2/\epsilon^2) \tau_n^{1/2} |\Omega|^{1/2} \left(\int_{t^{n-1}}^{t^n} \|\phi_h u_h\|_{L^2(\Omega)}^2 dt \right)^{1/2} \\ & \leq (2\delta_1/\epsilon^2) \tau_n |\Omega| + (1/2\delta_1\epsilon^2) \int_{t^{n-1}}^{t^n} \|\phi_h u_h\|_{L^2(\Omega)}^2 dt. \end{aligned}$$

Similarly, using Young's inequality with appropriate $\delta_2 > 0$, we obtain

$$\int_{t^{n-1}}^{t^n} |\langle f, \phi_h \rangle| dt \leq (\delta_2/\epsilon^2) \int_{t^{n-1}}^{t^n} \|\phi_h\|_{H^1(\Omega)}^2 + (\epsilon^2/4\delta_2) \int_{t^{n-1}}^{t^n} \|f\|_{H^{-1}(\Omega)}^2 dt.$$

Substituting the last two inequalities into (3.6), summing the resulting inequalities and using the fact that $\phi_+^N \equiv 0$ (by definition) and rearranging terms, we obtain

$$\begin{aligned} \|u_h\|_{L^2[0,T;L^2(\Omega)]}^2 &\leq \|u_h^0\|_{L^2(\Omega)}\|\phi_{h+}^0\|_{L^2(\Omega)} + (\delta_2/\epsilon^2)\|\phi_h\|_{L^2[0,T;H^1(\Omega)]}^2 \\ &\quad + (\epsilon^2/4\delta_2)\|f\|_{L^2[0,T;H^{-1}(\Omega)]}^2 + (2\delta_1/\epsilon^2)\sum_{n=1}^N \tau_n|\Omega| + (1/2\delta_1\epsilon^2)\|\phi_h u_h\|_{L^2[0,T;L^2(\Omega)]}^2 \\ &\leq (\delta_3/\epsilon^2)\|\phi_{h+}^0\|_{L^2(\Omega)}^2 + (\epsilon^2/4\delta_3)\|u_h^0\|_{L^2(\Omega)}^2 + (\delta_2/\epsilon^2)\|\phi_h\|_{L^2[0,T;H^1(\Omega)]}^2 \\ &\quad + (\epsilon^2/4\delta_2)\|f\|_{L^2[0,T;H^{-1}(\Omega)]}^2 + (2\delta_1/\epsilon^2)\sum_{n=1}^N \tau_n|\Omega| + (1/2\delta_1\epsilon^2)\|\phi_h u_h\|_{L^2[0,T;L^2(\Omega)]}^2. \end{aligned}$$

Using the previous bounds on $\|\phi_{h+}^0\|_{L^2(\Omega)}$, $\|\phi_h\|_{L^2[0,T;H^1(\Omega)]}$, $(1/\epsilon)\|\phi_h\|_{L^2[0,T;L^2(\Omega)]}$, and $(1/\epsilon)\|\phi_h u_h\|_{L^2[0,T;L^2(\Omega)]}$, in terms of $\|u_h\|_{L^2[0,T;L^2(\Omega)]}$ via (3.3) and choosing $\delta_1 = 2\epsilon^2$, $\delta_2 = \delta_3 = 1/4$, to hide the resulting terms on the left, we obtain,

$$\|u_h\|_{L^2(0,T;L^2(\Omega))} \leq C \left(T^{1/2} + \epsilon (\|u_h^0\|_{L^2(\Omega)} + \|f\|_{L^2[0,T;H^{-1}(\Omega)]}) \right).$$

with C an algebraic constant, depending only upon $|\Omega|$. Setting $w_h = u_h$, in (3.1) respectively and using the Poincaré, and Young's inequalities we obtain:

$$\begin{aligned} &(1/2)\|u_{h-}^n\|_{L^2(\Omega)}^2 - (1/2)\|u_{h-}^{n-1}\|_{L^2(\Omega)}^2 + (1/2)\|u_{h-}^{n-1}\|_{L^2(\Omega)}^2 \\ &\quad + \int_{t^{n-1}}^{t^n} \left((C/2)\|u_h\|_{H^1(\Omega)}^2 + (1/\epsilon^2)\|u_h\|_{L^4(\Omega)}^4 \right) dt \\ (3.7) \quad &\leq (1/\epsilon^2) \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)}^2 dt + \int_{t^{n-1}}^{t^n} (1/C)\|f\|_{H^{-1}(\Omega)}^2 dt. \end{aligned}$$

The second estimate follows by summation and the previously developed estimate on $L^2[0,T;L^2(\Omega)]$. \square

We close this subsection by a short remark.

Remark 3.3. It is evident that the key estimate with respect the dependence upon $(1/\epsilon)$ concerns the term $(1/\epsilon^2) \int_{t^{n-1}}^{t^n} \int_{\Omega} u_h w_h dx dt$ which has the wrong sign and not the term $(1/\epsilon^2) \int_{t^{n-1}}^{t^n} \int_{\Omega} u_h^3 w_h dx dt$ which is positive when setting $w_h = u_h$. For this reason the estimate of (3.1) does not lead to an estimate, with bounds independent of $\exp(1/\epsilon)$ when using Gronwall type arguments even for the lowest order scheme. To the contrary the duality argument of Lemma 3.2, leads to polynomial dependence upon $(1/\epsilon)$, without imposing any condition between τ, h , and under minimal regularity assumptions. The key question regarding the stability at arbitrary time-points, i.e. in $L^\infty[0,T;L^2(\Omega)]$, will be considered next.

4. ESTIMATES AT ARBITRARY TIME-POINTS AND CONVERGENCE UNDER MINIMAL REGULARITY

We will employ the theory of the approximation of the discrete characteristic functions (see e.g. [8, 9, 31]), which was used to develop estimates at arbitrary time points for a general class linear parabolic pdes and for the Navier-Stokes respectively. The main advantage of this approach is that the proof does not need any additional regularity, apart from the one needed to guarantee the existence of a weak solution, i.e., we do not assume that $u_t \in L^2[0,T;L^2(\Omega)]$ which is frequently used in the literature for dG approximations of parabolic pdes. In addition, we

will be able to obtain stability estimates without assuming any explicit dependence upon τ and h . A key feature of our analysis is that we are able to include high order schemes.

4.1. Preliminaries: Approximation of discrete characteristic functions.

Ideally, to obtain a stability estimate at arbitrary $t \in (t^{n-1}, t^n]$, we would like to substitute $u_h = \chi_{[t^{n-1}, t)} u_h$ into the discrete equations (3.1). However, this choice is not available in the discrete setting, since $\chi_{[t^{n-1}, t)} u_h$ is not a member of \mathcal{U}_h , unless t coincides with a partition point. Therefore, approximations of such functions need to be constructed. This is done in [8, Section 2.3]. For completeness we state the main results. The approximations are constructed on the interval $(0, \tau)$, and they are invariant under translations. For fixed (but arbitrary) $t \in (0, \tau)$ let $p \in \mathcal{P}_k(0, \tau)$, and denote the discrete approximation of $\chi_{[0, t)} p$ by the polynomial $\tilde{p} \in \mathcal{P}_k(0, \tau)$ with, $\tilde{p}(0) = p(0)$ which satisfies

$$\int_0^\tau \tilde{p} q = \int_0^t p q \quad \forall q \in \mathcal{P}_{k-1}(0, \tau).$$

To motivate the above construction we simply observe that for $q = p'$ we obtain $\int_0^\tau p' \tilde{p} = \int_0^t p p' = \frac{1}{2}(p^2(t) - p^2(0))$.

It is clear that this construction can be extended to approximations of $\chi_{[0, t)} u$ for $u \in \mathcal{P}_k[0, \tau; U]$ where U is a linear space. Note that if $u \in \mathcal{P}_k[0, \tau; U]$ then it can be written as $u = \sum_{i=0}^k p_i(t) u_i$ where $p_i \in \mathcal{P}_k[0, \tau]$ and $u_i \in U$. The discrete approximation of $\chi_{[0, t)} u$ in $\mathcal{P}_k[0, \tau; U]$ is then defined by $\tilde{u} = \sum_{i=0}^k \tilde{p}_i(t) u_i$ and if U is a semi-inner product space we deduce,

$$\tilde{u}(0) = u(0), \quad \text{and} \quad \int_0^\tau (\tilde{u}, w)_U = \int_0^t (u, w)_U \quad \forall w \in \mathcal{P}_{k-1}[0, \tau; U].$$

It remains to quote the main results from [8, 9, 31].

Proposition 4.1. Suppose that U is a (semi) inner product space. Then the mapping $\sum_{i=0}^k p_i(t) u_i \rightarrow \sum_{i=0}^k \tilde{p}_i(t) u_i$ on $\mathcal{P}_k[0, \tau; U]$ is continuous in $\|\cdot\|_{L^2[0, \tau; U]}$. In particular,

$$\|\tilde{u}\|_{L^2[0, \tau; U]} \leq C_k \|u\|_{L^2[0, \tau; U]}, \quad \|\tilde{u} - \chi_{[0, t)} u\|_{L^2[0, \tau; U]} \leq C_k \|u\|_{L^2[0, \tau; U]}$$

where C_k is a constant depending on k .

Proof. See [8, Lemma 2.4]. □

A standard calculation gives an explicit formula of $\tilde{u} = \rho(s)z$, when we choose $u(s) = z \in U$ to be constant (see e.g. [9]).

Lemma 4.2. Fix $t \in [0, \tau]$ and let $\rho \in \mathcal{P}_k[0, \tau]$ characterized by

$$\rho(0) = 1, \quad \int_0^\tau \rho q = \int_0^t q, \quad q \in \mathcal{P}_{k-1}[0, \tau].$$

Then,

$$\rho(s) = 1 + (s/\tau) \sum_{i=0}^{k-1} c_i \hat{p}_i(s/\tau), \quad c_i = \int_{t/\tau}^1 \hat{p}_i(\eta) d\eta,$$

where $\{\hat{p}_i\}_{i=0}^{k-1}$ is an orthonormal basis of $\mathcal{P}_{k-1}[0, 1]$ in the (weighted) space $L_w^2[0, 1]$ having inner product

$$(\hat{p}, \hat{q}) = \int_0^1 \eta \hat{p}(\eta) \hat{q}(\eta) d\eta.$$

In particular, $\|\rho\|_{L^\infty(0, \tau)} \leq C_k$, where C_k is independent of $t \in [0, \tau]$.

4.2. The main stability estimate at arbitrary time points. Now, we are ready to state the main stability result at arbitrary time-points which plays a key role to the derivation of best approximation estimates. We emphasize that the time-discretization parameter τ is chosen independent of h and the dependence of the stability constant upon $1/\epsilon$ is polynomial.

Proposition 4.3. Suppose that $f \in L^2[0, T; H^{-1}(\Omega)]$, $u_0 \in L^2(\Omega)$, and let u_h be the approximate solution computed by using the discontinuous time-stepping scheme. Then, there exists constant C depending on Ω , C_k and T (but not ϵ), such that

$$\|u_h\|_{L^\infty[0, T; L^2(\Omega)]} \leq C(1/\epsilon).$$

Proof. Recall that setting $w_h = u_h$, in (3.1), using Poincaré and Young's inequality, we obtain respectively

$$\begin{aligned} & (1/2)\|u_{h-}^n\|_{L^2(\Omega)}^2 - (1/2)\|u_{h-}^{n-1}\|_{L^2(\Omega)}^2 + (1/2)\|[u_h^{n-1}]\|_{L^2(\Omega)}^2 \\ & + \int_{t^{n-1}}^{t^n} \left((C/2)\|u_h\|_{H^1(\Omega)}^2 + (1/\epsilon^2)\|u_h\|_{L^4(\Omega)}^4 \right) dt \\ & \leq (1/\epsilon^2) \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)}^2 dt + \int_{t^{n-1}}^{t^n} (2/C)\|f\|_{H^{-1}(\Omega)}^2 dt. \end{aligned}$$

In order to avoid the use of a Grönwall type argument, we will need to estimate the term $(1/\epsilon^2) \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)}^2$ using the approximation of the discrete characteristic. We employ properties of the discrete characteristic and its approximation by following the technique of [9] and the stability estimates of Lemma 3.2. For fixed $t \in [t^{n-1}, t^n]$ and $z_h \in U_h$ we substitute $w_h(s) = z_h \rho(s)$ into (3.1), where $\rho(s) \in \mathcal{P}_k[t^{n-1}, t^n]$ is constructed similar to Lemma 4.2, i.e.,

$$\rho(t^{n-1}) = 1, \quad \int_{t^{n-1}}^{t^n} \rho q = \int_{t^{n-1}}^t q, \quad q \in \mathcal{P}_{k-1}[t^{n-1}, t^n].$$

Recall that Lemma 4.2 asserts that $\|\rho\|_{L^\infty(t^{n-1}, t^n)} \leq C_k$, with C_k independent of t . Now, it is easy to see that with this particular choice of w_h ,

$$\begin{aligned} & \int_{t^{n-1}}^{t^n} (u_{ht}, w_h) ds + (u_{h+}^{n-1} - u_{h-}^{n-1}, w_{h+}^{n-1}) \\ & = \int_{t^{n-1}}^t (u_{ht}, z_h) ds + (u_{h+}^{n-1} - u_{h-}^{n-1}, \rho(t^{n-1}) z_h) = (u_h(t) - u_{h-}^{n-1}, z_h). \end{aligned}$$

Hence integrating by parts (in time) equation (3.1) and using the above computation, we obtain

$$\begin{aligned}
& (u_h(t) - u_{h-}^{n-1}, z_h) \\
&= - \int_{t^{n-1}}^{t^n} (a(u_h, z_h \rho) + (1/\epsilon^2)(u_h^3 - u_h, z_h \rho)) ds + \int_{t^{n-1}}^{t^n} \langle f, z_h \rho \rangle ds \\
&\leq C_k \left[\int_{t^{n-1}}^{t^n} \|\nabla u_h\|_{L^2(\Omega)} \|\nabla z_h\|_{L^2(\Omega)} + \int_{t^{n-1}}^{t^n} \|f\|_{H^{-1}(\Omega)} \|z_h\|_{H^1(\Omega)} ds \right. \\
&\quad \left. + (1/\epsilon^2) \int_{t^{n-1}}^{t^n} (\|u_h^3\|_{L^{4/3}(\Omega)} \|z_h\|_{L^4(\Omega)} + \|u_h\|_{L^2(\Omega)} \|z_h\|_{L^2(\Omega)}) ds \right],
\end{aligned}$$

where we have used Lemma 4.2 to bound $\|\xi\|_{L^\infty(t^{n-1}, t^n)} \leq C_k$ with C_k denoting a constant depending only on k, Ω . Note also that $z_h \in U_h$ (independent of s), hence the above inequality leads to

$$\begin{aligned}
(u_h(t) - u_{h-}^{n-1}, z_h) &\leq C_k \left[\int_{t^{n-1}}^{t^n} (\|u_h\|_{H^1(\Omega)} + \|f\|_{H^{-1}(\Omega)}) ds \right] \|z_h\|_{H^1(\Omega)} \\
&+ C_k (1/\epsilon^2) \left(\left[\int_{t^{n-1}}^{t^n} \|u_h\|_{L^4(\Omega)}^3 ds \right] \|z_h\|_{L^4(\Omega)} + \left[\int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)} ds \right] \|z_h\|_{L^2(\Omega)} \right).
\end{aligned}$$

Here we have used the fact $\|u_h^3\|_{L^{4/3}(\Omega)} = \|u_h\|_{L^4(\Omega)}^3$. Setting $z_h = u_h(t)$ (for the previously fixed $t \in [t^{n-1}, t^n]$), using Hölder's inequality, and integrating in time the resulting inequality, we obtain,

$$\begin{aligned}
& \int_{t^{n-1}}^{t^n} \|u_h(t)\|_{L^2(\Omega)}^2 dt \leq \|u_{h-}^{n-1}\|_{L^2(\Omega)} \tau_n^{1/2} \|u_h(t)\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \\
& + C_k \tau_n^{1/2} \left(\|u_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} + \|f\|_{L^2[t^{n-1}, t^n; H^{-1}(\Omega)]} \right) \int_{t^{n-1}}^{t^n} \|u_h(t)\|_{H^1(\Omega)} dt \\
& + C_k \tau_n^{1/4} (1/\epsilon^2) \left(\|u_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^3 \right) \int_{t^{n-1}}^{t^n} \|u_h(t)\|_{L^4(\Omega)} dt \\
(4.1) \quad & + C_k \tau_n^{1/2} (1/\epsilon^2) \|u_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \int_{t^{n-1}}^{t^n} \|u_h(t)\|_{L^2(\Omega)} dt.
\end{aligned}$$

Hölder's inequality implies that $\int_{t^{n-1}}^{t^n} \|u_h\|_{L^4(\Omega)} dt \leq \tau_n^{3/4} \|u_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}$, and $\int_{t^{n-1}}^{t^n} \|u_h\|_{H^1(\Omega)} dt \leq \tau_n^{1/2} \|u_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]}$. Therefore, using Young's inequalities we deduce (with different C_k),

$$\begin{aligned}
& (1/2) \int_{t^{n-1}}^{t^n} \|u_h(t)\|_{L^2(\Omega)}^2 dt \leq (\tau_n/2) \|u_{h-}^{n-1}\|_{L^2(\Omega)}^2 \\
& + C_k \tau_n \left(\|u_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]}^2 + \|f\|_{L^2[t^{n-1}, t^n; H^{-1}(\Omega)]}^2 \right) \\
(4.2) \quad & + C_k \tau_n (1/\epsilon^2) \left(\|u_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^4 + \|u_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]}^2 \right).
\end{aligned}$$

Now, using an inverse estimate, $\|u_h(t)\|_{L^2(\Omega)}^2 \leq (C_k/\tau_n) \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)}^2$, we obtain,

$$\begin{aligned}
\|u_h(t)\|_{L^2(\Omega)}^2 &\leq C_k \left[\|u_{h-}^{n-1}\|_{L^2(\Omega)}^2 + \|u_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]}^2 + \|f\|_{L^2[t^{n-1}, t^n; H^{-1}(\Omega)]}^2 \right. \\
&\quad \left. + (1/\epsilon^2) \left(\|u_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^4 + \|u_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]}^2 \right) \right].
\end{aligned}$$

The proof now follows by simply substituting the previously developed bounds of (3.2). \square

Remark 4.4. The above theorem states that the discontinuous Galerkin discretization inherits the stability estimates of the weak formulation under minimal regularity assumptions on the given data. This is an important asset related to the discontinuous (in time) Galerkin formulation.

4.3. Convergence under minimal regularity assumptions. We quote a discrete compactness argument of Walkington (see [31, Theorem 3.1]) which allows to recover strong convergence in an appropriate norm, and pass the limit through the nonlinear term. The compactness argument combined with the stability estimates of Lemma 3.2 and Proposition 4.3, imply the convergence of the space-time approximations under minimal regularity assumptions, while the dependence upon $(1/\epsilon)$ does not deteriorate any further.

The compactness argument concerns numerical approximations of solutions $u : [0, T] \rightarrow X$ of general evolution equations of the form

$$(4.3) \quad u_t + A(u) = f(u), \quad u(0) = u_0,$$

where X is a Banach space and each term of the equation takes values in X^* . Both $A(u) = A(t, u)$ and $f(u) = f(t, u)$ may depend upon t and are allowed to be nonlinear, however, in our setting only $f(u) \equiv -(1/\epsilon^2)(u^3 - u)$ contains nonlinear terms. Suppose that $X \subset H \subset X^*$ (with continuous embeddings) form the standard evolution triple, i.e., the pivot space H is a Hilbert space. The numerical schemes approximate the weak form of (4.3), i.e.,

$$(4.4) \quad \langle u_t, w \rangle + a(u, w) = \langle f(u), w \rangle, \quad \forall w \in X$$

where $a : X \times X \rightarrow \mathbb{R}$ is defined by $a(u, w) = (A(u), w)$. Set $F(u) \equiv f(u) - A(u)$. Then the following theorem [31, Theorem 3.1] establishes the compactness property of the discrete approximation.

Theorem 4.5. Let H be a Hilbert space, X be a Banach space and $X \subset H \subset X^*$ be dense and compact embeddings. Fix an integer $k \geq 0$ and let $1 \leq p, q < \infty$. Let $h > 0$ be the mesh parameter, and let $\{t^i\}_{i=0}^N$ denote a quasi-uniform partition of $[0, T]$. Assume that

- (1) For each $h, \tau > 0$, $u_h \in \{u_h \in L^p[0, T; U] \mid u_h|_{(t^{n-1}, t^n)} \in \mathcal{P}_k[t^{n-1}, t^n; U_h]\}$ and on each interval, satisfies

$$\int_{t^{n-1}}^{t^n} \langle u_{ht}, w_h \rangle dt + (u_{h+}^{n-1} - u_{h-}^{n-1}, w_{h+}^{n-1}) = \int_{t^{n-1}}^{t^n} \langle F(u_h), w_h \rangle dt$$

for every $w_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h]$.

- (2) $\{u_h\}_{h>0}$ is bounded in $L^p[0, T; U]$ and $\{\|F(u_h)\|_{L^q[0, T; U^*]}\}_{h>0}$ is also bounded.

Then,

- (1) If $p > 1$ then $\{u_h\}_{h>0}$ is compact in $L^r[0, T; H]$ for $1 \leq r < 2p$.
- (2) If $1 \leq (1/p) + (1/q) < 2$, and $\sum_{i=1}^N \|u_h\|_H^2 < C$ is bounded independent of h , then $\{u_h\}_{h>0}$ is compact in $L^r[0, T; H]$ for $1 \leq r < 2/((1/p) + (1/q) - 1)$.

Proof. See [31, Theorem 3.1]. \square

We will utilize the above result to obtain strong convergence of the discrete Allen-Cahn equation to the continuous one. The lack of any meaningful regularity

for the discrete time derivative due to the presence of discontinuities, requires special attention since the classical Aubin-Lions compactness argument is not directly applicable.

Theorem 4.6. Suppose that $f \in L^2[0, T; H^{-1}(\Omega)]$, $u_0 \in L^2(\Omega)$, and let $\epsilon < 1$ be a given parameter. Let $\{t^i\}_{i=0}^N$ denote a quasi-uniform partition of $[0, T]$. Suppose that the assumptions of Proposition 4.3 hold, and let $\tau, h \rightarrow 0$. Then,

$$u_h \rightarrow u \text{ weakly in } L^2[0, T; H_0^1(\Omega)], \quad u_h \rightarrow u \text{ weakly-}^* \text{ in } L^\infty[0, T; L^2(\Omega)],$$

and

$$u_h \rightarrow u \quad \text{strongly in } L^2[0, T; L^2(\Omega)].$$

In addition u is a weak solution of the Allen-Cahn equation.

Proof. We follow the same arguments with [31, Section 6]. The stability estimates of Lemma 3.2 and Proposition 4.3, imply (passing to a subsequence if necessary) there exists u such that $u_h \rightarrow u$ weakly in $L^2[0, T; H_0^1(\Omega)]$ and weakly- * in $L^\infty[0, T; L^2(\Omega)]$. It remains to obtain strong convergence in $L^2[0, T; L^2(\Omega)]$. For this purpose, fix $X = H_0^1(\Omega)$, $H = L^2(\Omega)$, and $F(u) = \Delta u - (1/\epsilon^2)(u^3 - u) - f$. It is easy to show that $F(u_h) \in L^{4/3}[0, T; H^{-1}(\Omega)]$. Indeed, $u_h \in L^2[0, T; H_0^1(\Omega)] \cap L^4[0, T; L^4(\Omega)]$, and $u_h \in L^\infty[0, T; L^2(\Omega)]$ clearly imply that $u_h^3 \in L^{4/3}[0, T; H^{-1}(\Omega)]$ by using standard interpolation theorems. The remaining terms can be handled easily. Therefore, using the Theorem 4.5, we obtain the desired strong convergence in $L^2[0, T; L^2(\Omega)]$. Choose $w_h \in C[0, T; U_h] \cap \mathcal{U}_h$, with $w_h(T) = 0$. Then, summing equations (3.1) from $n = 1$ to $n = N$, we deduce that

$$\begin{aligned} (u_h(T), w_h(T)) + \int_0^T (-\langle u_h, w_{ht} \rangle + a(u_h, w_h) + (1/\epsilon^2)\langle u_h^3 - u_h, w_h \rangle) dt \\ = \int_0^T \langle f, w_h \rangle dt + (u^0, w_h(0)). \end{aligned}$$

Note that we may pass the limit through the linear terms due to the stability estimates on u_h and the fact that $w_h \in C[0, T; U_h] \cap \mathcal{U}_h$. The semi-linear term can be treated by the strong convergence on $L^2[0, T; L^2(\Omega)]$. Indeed, using Holder's inequality, Landyżeskaya-Gagliardo-Nirenberg interpolation inequality,

$$\begin{aligned} \int_0^T |\langle u_h^3 - u^3, w_h \rangle| dt &\leq \int_0^T |\langle (u_h - u)(u_h^2 + u^2 + u_h u), w_h \rangle| dt \\ &\leq C \int_0^T \|u_h - u\|_{L^3(\Omega)} (\|u_h\|_{L^4(\Omega)}^2 + \|u\|_{L^4(\Omega)}^2) \|w_h\|_{L^6(\Omega)} dt \\ &\leq C \|w_h\|_{C[0, T; H^1(\Omega)]} \int_0^T \|u_h - u\|_{L^2(\Omega)}^{1/2} \|u_h - u\|_{H^1(\Omega)}^{1/2} (\|u_h\|_{L^4(\Omega)}^2 + \|u\|_{L^4(\Omega)}^2) dt \\ &\leq C \|w_h\|_{C[0, T; H^1(\Omega)]} \|u_h - u\|_{L^2[0, T; L^2(\Omega)]}^{1/2} \|u_h - u\|_{L^2[0, T; H^1(\Omega)]}^{1/2} \\ &\quad \times (\|u_h\|_{L^4[0, T; L^4(\Omega)]} + \|u\|_{L^4[0, T; L^4(\Omega)]})^2. \end{aligned}$$

A standard density argument, now completes the proof. \square

5. ERROR ESTIMATES

5.1. Projections. The following projections related to discontinuous Galerkin time-stepping schemes will be used.

Definition 5.1. (1) The projection $P_n^{loc} : C[t^{n-1}, t^n; L^2(\Omega)] \rightarrow \mathcal{P}_k[t^{n-1}, t^n; U_h]$ satisfies $(P_n^{loc} w)^n = P_h w(t^n)$, and

$$\int_{t^{n-1}}^{t^n} (w - P_n^{loc} w, W_h) = 0, \quad \forall W_h \in \mathcal{P}_{k-1}[t^{n-1}, t^n; U_h].$$

In the above definition, we have used the convention $(P_n^{loc} w)^n \equiv (P_n^{loc} w)(t^n)$, and $P_h : L^2(\Omega) \rightarrow U_h$ is the orthogonal projection operator onto $U_h \subset H^1(\Omega)$.

(2) The projection $P_h^{loc} : C[0, T; L^2(\Omega)] \rightarrow \mathcal{U}_h$ satisfies

$$P_h^{loc} w \in \mathcal{U}_h \text{ and } (P_h^{loc} w)|_{(t^{n-1}, t^n]} = P_n^{loc}(w|_{[t^{n-1}, t^n]}).$$

In the following Lemma, we collect several results regarding (optimal) rates of convergence for the above projection (see e.g. [9]).

Lemma 5.2. Let $U_h \subset H^1(\Omega)$, and P_h^{loc} defined in Definition 5.1 respectively. Then, for all $w \in L^2[0, T; H^{l+1}(\Omega)] \cap H^{k+1}[0, T; L^2(\Omega)]$ there exists constant $C \geq 0$ independent of h, τ such that

$$\|w - P_h^{loc} w\|_{L^2[0, T; L^2(\Omega)]} \leq C(h^{l+1} \|w\|_{L^2[0, T; H^{l+1}(\Omega)]} + \tau^{k+1} \|w^{(k+1)}\|_{L^2[0, T; L^2(\Omega)]}).$$

$$\|w - P_h^{loc} w\|_{L^2[0, T; H^1(\Omega)]} \leq C(h^l \|w\|_{L^2[0, T; H^{l+1}(\Omega)]} + (\tau^{k+1}/h) \|w^{(k+1)}\|_{L^2[0, T; L^2(\Omega)]}).$$

$$\|w - P_h^{loc} w\|_{L^\infty[0, T; L^2(\Omega)]} \leq C(h^{l+1} \|w\|_{L^\infty[0, T; H^{l+1}(\Omega)]} + \tau^{k+1} \|w^{(k+1)}\|_{L^\infty[0, T; L^2(\Omega)]}).$$

Let $k = 0, l = 1$, and $w \in L^2[0, T; H^2(\Omega)] \cap H^1[0, T; L^2(\Omega)]$. Then there exists constant $C \geq 0$ independent of h, τ such that,

$$\begin{aligned} & \|w - P_h^{loc} w\|_{L^\infty[0, T; L^2(\Omega)]} + \|w - P_h^{loc} w\|_{L^2[0, T; H^1(\Omega)]} \leq C(h \|w\|_{L^2[0, T; H^2(\Omega)]} \\ & + \tau^{1/2} (\|w_t\|_{L^2[0, T; L^2(\Omega)]} + \|w\|_{L^2[0, T; H^2(\Omega)]})). \end{aligned}$$

Remark 5.3. If more regularity (in time) is available then the above estimates can be improved. In particular, if $w \in L^2[0, T; H^{l+1}(\Omega)] \cap H^{k+1}[0, T; H^1(\Omega)]$, then we obtain,

$$\|w - P_h^{loc} w\|_{L^2[0, T; H^1(\Omega)]} \leq C(h^l \|w\|_{L^2[0, T; H^{l+1}(\Omega)]} + \tau^{k+1} \|w^{(k+1)}\|_{L^2[0, T; H^1(\Omega)]}).$$

The fully-discrete Galerkin orthogonality can be written as follows: Subtracting (3.1) from (2.1), we obtain for every $w_h \in \mathcal{U}_h$ and for $n = 1, \dots, N$,

$$\begin{aligned} (5.1) \quad & (e_-^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} (-\langle e, w_{ht} \rangle + a(e, w_h)) dt \\ & + (1/\epsilon^2) \int_{t^{n-1}}^{t^n} ((u_h^3 - u^3, w_h) - (u_h - u, w_h)) dt = (e_-^{n-1}, w_{h+}^{n-1}) \end{aligned}$$

where $e = u_h - u$ denotes the error. We will split the error as $e = (u_h - u_p) + (u_p - u) \equiv e_h + e_p$, where u_p is the discontinuous Galerkin solution of a linear parabolic pde with right hand side $u_t - \Delta u$, and initial data $u_{p0} = P_h u_0$, i.e., for every $w_h \in \mathcal{U}_h$ and for $n = 1, \dots, N$, $u_p \in \mathcal{U}_h$ is the solution of,

$$\begin{aligned} (5.2) \quad & (u_{p-}^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} (-\langle u_p, w_{ht} \rangle + a(u_p, w_h)) dt \\ & = (u_{p+}^{n-1}, w_{h+}^{n-1}) + \int_{t^{n-1}}^{t^n} \langle u_t - \Delta u, w_h \rangle dt. \end{aligned}$$

Integrating by parts the last term of the right hand side, we obtain the orthogonality condition: For $n = 1, \dots, N$, and $w_h \in \mathcal{U}_h$

$$(5.3) \quad (e_{p-}^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} \left(-\langle e_p, w_{ht} \rangle + a(e_p, w_h) \right) dt = (e_{p+}^{n-1}, w_{+}^{n-1}).$$

The following best approximation estimates under minimal regularity assumptions that bound the error $e_p = u_p - u$ in terms of the local projections of Definition 5.1 are straightforward application of [8, Theorem 2.2 and Theorem 2.3]).

$$(5.4) \quad \|e_p\|_{L^\infty[0,T;L^2(\Omega)]} + \|e_p\|_{L^2[0,T;H^1(\Omega)]} \leq C \left(\|P_h u(0) - u(0)\|_{L^2(\Omega)} + \|u - P_h^{loc} u\|_{L^\infty[0,T;L^2(\Omega)]} + \|u - P_h^{loc} u\|_{L^2[0,T;H^1(\Omega)]} \right),$$

where C is a constant depending upon Ω and the constant C_k of Proposition 4.1. In addition,

$$(5.5) \quad \|u_p\|_{L^\infty[0,T;H^1(\Omega)]} \leq C(\|u_0\|_{H^1(\Omega)} + \|u_t - \Delta u\|_{L^2[0,T;L^2(\Omega)]}).$$

by [9, Theorem 4.10]. Returning back to the orthogonality condition (5.1) and using (5.3) we obtain, the following relation for $e_h = u_h - u_p$: For all $w_h \in \mathcal{U}_h$ and for $n = 1, \dots, N$,

$$(5.6) \quad (e_{h-}^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} (-\langle e_h, w_{ht} \rangle + a(e_h, w_h)) dt + (1/\epsilon^2) \int_{t^{n-1}}^{t^n} ((u_h^3 - u^3, w_h) - (u_h - u, w_h)) dt = (e_{h-}^{n-1}, w_{h+}^{n-1}).$$

Adding and subtracting the term u_p^3 in the nonlinear term, we equivalently obtain,

$$(5.7) \quad \begin{aligned} & (e_{h-}^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} (-\langle e_h, w_{ht} \rangle + a(e_h, w_h)) dt - (e_{h-}^{n-1}, w_{h+}^{n-1}) \\ & + (1/\epsilon^2) \int_{t^{n-1}}^{t^n} ((e_h(u_h^2 + u_p^2 + u_h u_p), w_h) - (e_h, w_h)) dt \\ & = (1/\epsilon^2) \int_{t^{n-1}}^{t^n} ((e_p(u_p^2 + u^2 + u_p u), w_h) - (e_p, w_h)) dt. \end{aligned}$$

Our focus is to bound e_h in terms of e_p without introducing constants that depend exponentially upon $1/\epsilon$. The main strategy is to bound $\|e_h\|_{L^2[0,T;L^2(\Omega)]}$ in terms of norms of e_p and with constants that do not depend exponentially upon $1/\epsilon$. For this purpose, we employ duality.

5.2. An auxiliary duality argument. We follow the approach presented in Section 3. In particular, given right hand side $e_h \in L^\infty[0,T;L^2(\Omega)]$, and terminal data $\psi_{h+}^N = 0$, we seek $\psi_h \in \mathcal{U}_h$ such that for all $w_h \in \mathcal{P}_k[t^{n-1}, t^n; \mathcal{U}_h]$, and for all $n = N, \dots, 1$,

$$(5.8) \quad \begin{aligned} & -(\psi_{h+}^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} ((\psi_h, w_{ht}) + a(\psi_h, w_h)) dt + (\psi_{h+}^{n-1}, w_{h+}^{n-1}) \\ & + (1/\epsilon^2) \int_{t^{n-1}}^{t^n} (\psi_h, v_h) dt = \int_{t^{n-1}}^{t^n} (e_h, w_h) dt. \end{aligned}$$

Note that (5.8) is a linear parabolic pde, with zero terminal data but with $\frac{1}{\epsilon^2}(\cdot, \cdot)$ term with positive sign. The above formulation is the dG time stepping discretization of the following problem: For any $w \in L^2[0, T; H^1(\Omega)] \cap H^1[0, T; H^{-1}(\Omega)]$, we seek $\psi \in L^2[0, T; H_0^1(\Omega)] \cap H^1[0, T; H^{-1}(\Omega)]$, such that:

$$(5.9) \quad \int_0^T ((\psi, w_t) + a(\psi, w))dt + (\psi(0), w(0)) + (1/\epsilon^2) \int_0^T (\psi, w)dt = \int_0^T (e_h, w)dt.$$

Using standard energy arguments, we deduce stability estimates in various norms.

$$\begin{aligned} \|\psi\|_{L^\infty[0, T; L^2(\Omega)]} + \|\psi\|_{L^2[0, T; H^1(\Omega)]} + (1/\epsilon)\|\psi\|_{L^2[0, T; L^2(\Omega)]} &\leq C\epsilon\|e_h\|_{L^2[0, T; L^2(\Omega)]}, \\ \|\psi_t\|_{L^2[0, T; L^2(\Omega)]} + \|\psi\|_{L^2[0, T; H^2(\Omega)]} &\leq C\|e_h\|_{L^2[0, T; L^2(\Omega)]}. \end{aligned}$$

We split the error as $\psi_h - \psi = d_h + d_p$, where $d_h = \psi_h - P_h^{loc}\psi$ and $d_p = P_h^{loc}\psi - \psi$. Here (abusing the notation) we denote by P_h^{loc} the standard discontinuous (in time) Galerkin projection of Definition 5.1 suitably modified to handle the backwards in time problem. We can write the orthogonality condition, as follows:

$$\begin{aligned} &-(d_{h+}^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} ((d_h, w_{ht}) + a(d_h, w_h))dt + (d_{h+}^{n-1}, w_{h+}^{n-1}) \\ &+ (1/\epsilon^2) \int_{t^{n-1}}^{t^n} (d_h, w_h)dt = \int_{t^{n-1}}^{t^n} (e_h, w_h)dt - \int_{t^{n-1}}^{t^n} a(d_p, w_h)dt \\ (5.10) \quad &-(1/\epsilon^2) \int_{t^{n-1}}^{t^n} (d_p, v_h)dt. \end{aligned}$$

Note that by the definition of the projection P_h^{loc} , we have that $\int_{t^{n-1}}^{t^n} (d_p, w_{ht})dt = 0$, $(d_{p+}^n, w_{h-}^n) = 0$, and $(d_{p+}^{n-1}, w_{h+}^{n-1}) = 0$. We can prove the following estimates by standard arguments:

Lemma 5.4. Suppose that $\psi, \psi_h \in$ satisfy (5.9) and (5.8) respectively. Then, under the assumptions of Lemma 3.2, there exists a constant C independent of ϵ , such that

$$\begin{aligned} \|d_h\|_{L^2[0, T; H^1(\Omega)]} + \frac{1}{\epsilon}\|d_h\|_{L^2[0, T; L^2(\Omega)]} &\leq C(\|d_p\|_{L^2[0, T; H^1(\Omega)]} + \frac{1}{\epsilon}\|d_p\|_{L^2[0, T; L^2(\Omega)]}), \\ \|d_h\|_{L^\infty[0, T; L^2(\Omega)]} &\leq C(\|d_p\|_{L^2[0, T; H^1(\Omega)]} + \frac{1}{\epsilon}\|d_p\|_{L^2[0, T; L^2(\Omega)]}), \\ \|d_h\|_{L^2[0, T; L^2(\Omega)]} &\leq C((\tau^{1/2} + h)\|d_p\|_{L^2[0, T; H^1(\Omega)]} + \|d_p\|_{L^2[0, T; L^2(\Omega)]}). \end{aligned}$$

In addition, there exists a constant C depending only upon the domain and the polynomial degree, such that

$$\|d_h\|_{L^\infty[0, T; H^1(\Omega)]} \leq C\|e_h\|_{L^2[0, T; L^2(\Omega)]}.$$

Proof. Setting $w_h = d_h$ into (5.10), and using standard algebra, we obtain the first estimate. For the estimate at arbitrary time-points we follow the arguments of [8, Theorem 2.3]. In order to obtain the improved rate for the $L^2[0, T; L^2(\Omega)]$ norm we employ a duality argument to derive a better bound for the quantity $\|d_h\|_{L^2[0, T; L^2(\Omega)]}^2$. For this purpose, we define a forward in time parabolic problem with right hand side $d_h \in L^2[0, T; L^2(\Omega)]$ and zero initial data. For $n = 1, \dots, N$ and for all $w \in L^2[0, T; H^1(\Omega)] \cap H^1[0, T; H^1(\Omega)^*]$, we seek $\xi \in L^2[0, T; H_0^1(\Omega)] \cap$

$H^1[0, T; H^{-1}(\Omega)]$, such that

$$(5.11) \quad (\xi(T), w(T)) + \int_0^T (-\langle \xi, w_t \rangle + a(w, \xi) + \frac{1}{\epsilon^2}(\xi, w)) dt = \int_0^T (d_h, w) dt.$$

We note that we impose zero initial data, i.e., $\xi(0) \equiv 0$. It is clear that the following stability estimates hold: There exists a constant C independent of τ, h and ϵ such that,

$$(5.12) \quad \begin{aligned} & \|\xi\|_{L^\infty[0, T; L^2(\Omega)]} + \|\xi\|_{L^2[0, T; H^1(\Omega)]} + (1/\epsilon)\|\xi\|_{L^2[0, T; L^2(\Omega)]} \leq C\epsilon\|d_h\|_{L^2[0, T; L^2(\Omega)]}, \\ & \|\xi\|_{L^2[0, T; H^2(\Omega)]} + \|\xi_t\|_{L^2[0, T; L^2(\Omega)]} + \|\xi\|_{L^\infty[0, T; H^1(\Omega)]} \leq C\|d_h\|_{L^2[0, T; L^2(\Omega)]}. \end{aligned}$$

Note that due to the presence of discontinuities on d_h , we can not improve regularity of ξ in time. The discontinuous time-stepping scheme can be defined as follows: Given, initial data $\xi_h^0 = 0$, we seek $\xi_h \in \mathcal{U}_h$ such that for all $w_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h]$,

$$(5.13) \quad \begin{aligned} & (\xi_{h-}^n, v_{h-}^n) + \int_{t^{n-1}}^{t^n} (-\langle \xi_h, w_{ht} \rangle + a(\xi_h, w_h) + \frac{1}{\epsilon^2}(\xi_h, w_h)) dt \\ & - (\xi_{h-}^{n-1}, v_{h+}^{n-1}) = \int_{t^{n-1}}^{t^n} (d_h, v_h) dt. \end{aligned}$$

Setting $w_h = \xi_h$, we deduce the following stability bound:

$$(5.14) \quad \|\xi_h\|_{L^2[0, T; L^2(\Omega)]} \leq C\epsilon^2\|d_h\|_{L^2[0, T; L^2(\Omega)]}.$$

It is now clear that we have the following estimate for $\xi - \xi_h$, which is a straightforward application of the previous estimates in $L^2[0, T; H^1(\Omega)]$ (see e.g. [8, Theorem 2.2]), the approximation properties of projections P_h^{loc} , and (5.12),

$$\begin{aligned} & \|\xi - \xi_h\|_{L^2[0, T; H^1(\Omega)]} + \frac{1}{\epsilon}\|\xi - \xi_h\|_{L^2[0, T; L^2(\Omega)]} \\ & \leq C(\tau^{1/2} + h)(\|\xi\|_{L^2[0, T; H^2(\Omega)]} + \|\xi_t\|_{L^2[0, T; L^2(\Omega)]}) \leq C(\tau^{1/2} + h)\|d_h\|_{L^2[0, T; L^2(\Omega)]}, \end{aligned}$$

where the constant C is independent of τ, h, ϵ . We note that the lack of regularity on the right hand side, restricts the rate of convergence to the rate given by the lowest order scheme $l = 1, k = 0$, even if high order schemes (in time) are chosen. Setting $w_h = d_h$, into (5.13) and integrating by parts with respect to time we obtain,

$$(5.15) \quad \begin{aligned} & (\xi_{h+}^{n-1}, d_{h+}^{n-1}) + \int_{t^{n-1}}^{t^n} ((\xi_h, d_{ht}) + a(d_h, \xi_h) + \frac{1}{\epsilon^2}(\xi_h, d_h)) dt - (\xi_{h-}^{n-1}, d_{h+}^{n-1}) \\ & = \int_{t^{n-1}}^{t^n} \|d_h\|_{L^2(\Omega)}^2 dt. \end{aligned}$$

Setting $w_h = \xi_h$ into (5.10), we obtain:

$$(5.16) \quad \begin{aligned} & -(d_{h+}^n, \xi_{h-}^n) + \int_{t^{n-1}}^{t^n} ((d_h, \xi_{ht}) + a(d_h, \xi_h)) dt + (d_{h+}^{n-1}, \xi_{h+}^{n-1}) \\ & + (1/\epsilon^2) \int_{t^{n-1}}^{t^n} (d_h, \xi_h) dt \\ & = \int_{t^{n-1}}^{t^n} (e_h, \xi_h) dt - \int_{t^{n-1}}^{t^n} a(d_p, \xi_h) dt - (1/\epsilon^2) \int_{t^{n-1}}^{t^n} (d_p, \xi_h) dt. \end{aligned}$$

Subtracting (5.16) from (5.15) we obtain:

$$\begin{aligned}
(d_{h+}^n, \xi_{h-}^n) - (\xi_{h-}^{n-1}, d_{h+}^{n-1}) &= \int_{t^{n-1}}^{t^n} \|d_h\|_{L^2(\Omega)}^2 dt + \int_{t^{n-1}}^{t^n} (a(d_p, \xi_h) + \frac{1}{\epsilon^2}(d_p, \xi_h)) dt \\
&= \int_{t^{n-1}}^{t^n} \|d_h\|_{L^2(\Omega)}^2 dt + \int_{t^{n-1}}^{t^n} (a(d_p, \xi_h - \xi) + a(d_p, \xi) + \frac{1}{\epsilon^2}(d_p, \xi_h)) dt \\
&= \int_{t^{n-1}}^{t^n} \|d_h\|_{L^2(\Omega)}^2 dt + \int_{t^{n-1}}^{t^n} (a(d_p, \xi_h - \xi) - (d_p, \Delta \xi) + \frac{1}{\epsilon^2}(d_p, \xi_h)) dt
\end{aligned}$$

where at the last two equalities we have used integration by parts (in space). Then summing the above inequalities and using the fact that $d_{h+}^N \equiv 0$ and $\xi_h^0 = 0$ (by definition) and rearranging terms, we obtain

$$\begin{aligned}
(1/2)\|d_h\|_{L^2[0,T;L^2(\Omega)]}^2 &\leq C \int_0^T \|d_p\|_{L^2(\Omega)} \|\xi\|_{H^2(\Omega)} dt \\
&\quad + C \int_0^T (\|\xi_h - \xi\|_{H^1(\Omega)} \|d_p\|_{H^1(\Omega)} + \frac{1}{\epsilon^2} \|d_p\|_{L^2(\Omega)} \|\xi_h\|_{L^2(\Omega)}) dt \\
&\leq C \left(\|d_p\|_{L^2[0,T;L^2(\Omega)]} \|\xi\|_{L^2[0,T;H^2(\Omega)]} + \|\xi_h - \xi\|_{L^2[0,T;H^1(\Omega)]} \|d_p\|_{L^2[0,T;H^1(\Omega)]} \right. \\
&\quad \left. + \frac{1}{\epsilon^2} \|d_p\|_{L^2[0,T;L^2(\Omega)]} \|\xi_h\|_{L^2[0,T;L^2(\Omega)]} \right) \\
&\leq C \left(\|d_p\|_{L^2[0,T;L^2(\Omega)]} \|d_h\|_{L^2[0,T;L^2(\Omega)]} + (\tau^{1/2} + h) \|d_h\|_{L^2[0,T;L^2(\Omega)]} \|d_p\|_{L^2[0,T;H^1(\Omega)]} \right. \\
&\quad \left. + \frac{1}{\epsilon^2} \|d_p\|_{L^2[0,T;L^2(\Omega)]} \epsilon^2 \|d_h\|_{L^2[0,T;L^2(\Omega)]} \right).
\end{aligned}$$

Here, we have used the Cauchy-Schwarz inequality, the stability bounds of dual equation (5.12) and (5.14), and the error estimates on $\xi_h - \xi$. Finally, for the stability estimate we refer the reader to [9, Theorem 4.10]. \square

Now, we are ready to prove the following bound, which will allow us to apply a bootstrap argument. Using an appropriate duality argument, we avoid the use of Grönwall type inequalities.

Proposition 5.5. Suppose that $f \in L^2[0, T; L^2(\Omega)]$ and $u_0 \in H_0^1(\Omega)$. Let τ, h, ϵ , satisfy $\tau^{1/2} + h \leq \min\{C_0 \epsilon^2, \frac{1}{(\|u\|_{L^2[0,T;H^2(\Omega)]} + \|u_t\|_{L^2[0,T;L^2(\Omega)]})^2}\}$, with constant C_0 depending only upon the domain (independent of ϵ, h, τ). Then, there exists a constant $C > 0$ independent of τ, h, ϵ , such that following estimates hold:
For $d = 2$,

$$\begin{aligned}
\|e_h\|_{L^2[0,T;L^2(\Omega)]} &\leq C \left((\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \|e_p\|_{L^2[0,T;L^2(\Omega)]} \right. \\
&\quad \left. + \frac{1}{\epsilon^2} (\tau^{1/2} + h) (\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]} + \|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}) \right. \\
&\quad \left. + (\tau^{1/2} + h) \|e_h\|_{L^2[0,T;H^1(\Omega)]} \right).
\end{aligned}$$

For $d = 3$,

$$\begin{aligned} \|e_h\|_{L^2[0,T;L^2(\Omega)]} &\leq C \left((\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \|e_p\|_{L^2[0,T;L^2(\Omega)]} \right. \\ &\quad + \frac{1}{\epsilon^2} (\tau^{1/2} + h)^{3/4} (\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]} + \|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}) \\ &\quad \left. + (\tau^{1/2} + h) \|e_h\|_{L^2[0,T;H^1(\Omega)]} \right). \end{aligned}$$

Proof. Setting $w_h = e_h$ into (5.10), and using integration by parts in time, we obtain: For all $n = N, \dots, 1$,

$$\begin{aligned} &-(d_{h+}^n, e_{h-}^n) + (d_{h-}^n, e_{h-}^n) + \int_{t^{n-1}}^{t^n} -(d_{ht}, e_h) dt \\ &+ \int_{t^{n-1}}^{t^n} (a(d_h, e_h) dt + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} (d_h, e_h) dt \\ (5.17) \quad &= \int_{t^{n-1}}^{t^n} \|e_h\|_{L^2(\Omega)}^2 dt - \int_{t^{n-1}}^{t^n} a(d_p, e_h) dt - \int_{t^{n-1}}^{t^n} \frac{1}{\epsilon^2} (d_p, e_h) dt. \end{aligned}$$

Setting $w_h = d_h$ into (5.7), we deduce for all $n = 1, \dots, N$,

$$\begin{aligned} &(e_{h-}^n, d_{h-}^n) + \int_{t^{n-1}}^{t^n} (-(e_h, d_{ht}) + a(e_h, d_h)) dt - (e_{h-}^{n-1}, d_{h+}^{n-1}) \\ &+ \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} ((e_h(u_h^2 + u_p^2 + u_h u_p), d_h) - (e_h, d_h)) dt \\ (5.18) \quad &= \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} ((e_p(u_p^2 + u^2 + u_p u), d_h) - (e_p, d_h)) dt. \end{aligned}$$

Subtracting (5.18) from (5.17), and rearranging terms, we obtain, for all $n = 1, \dots, N$,

$$\begin{aligned} &\int_{t^{n-1}}^{t^n} \|e_h\|_{L^2(\Omega)}^2 dt = -(d_{h+}^n, e_{h-}^n) + (e_{h-}^{n-1}, d_{h+}^{n-1}) \\ &+ \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} ((e_p(u_p^2 + u^2 + u_p u), d_h) - (e_p, d_h)) dt \\ &+ \int_{t^{n-1}}^{t^n} a(d_p, e_h) dt - \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} ((u_h^2 + u_p^2 + u_h u_p) d_h, e_h) dt \\ (5.19) \quad &+ \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} (d_p, e_h) dt + \frac{2}{\epsilon^2} \int_{t^{n-1}}^{t^n} (d_h, e_h) dt. \end{aligned}$$

Summing the equalities (5.19), noting that $e_{h-}^0 = 0 = d_{h+}^N$, and using Hölder's and Young's inequalities, we obtain,

$$\begin{aligned}
\int_0^T \|e_h\|_{L^2(\Omega)}^2 &\leq \frac{C}{\epsilon^2} \int_0^T \|e_p\|_{L^2(\Omega)} (\|u_p^2\|_{L^3(\Omega)} + \|u^2\|_{L^3(\Omega)}) \|d_h\|_{L^6(\Omega)} dt \\
&\quad + \frac{1}{\epsilon^2} \int_0^T \|e_p\|_{L^2(\Omega)} \|d_h\|_{L^2(\Omega)} dt + \int_0^T \|d_p\|_{H^1(\Omega)} \|e_h\|_{H^1(\Omega)} dt \\
&\quad + \frac{C}{\epsilon^2} \int_0^T (\|e_h u_h\|_{L^2(\Omega)} + \|e_h u_p\|_{L^2(\Omega)}) \|d_h\|_{L^4(\Omega)} (\|u_h\|_{L^4(\Omega)} + \|u_p\|_{L^4(\Omega)}) dt \\
&\quad + \frac{1}{\epsilon^2} \int_0^T \|d_p\|_{L^2(\Omega)} \|e_h\|_{L^2(\Omega)} dt + \frac{2}{\epsilon^2} \int_0^T \|d_h\|_{L^2(\Omega)} \|e_h\|_{L^2(\Omega)} dt \\
&\leq \frac{C}{\epsilon^2} (\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \|e_p\|_{L^2[0,T;L^2(\Omega)]} \|d_h\|_{L^2[0,T;H^1(\Omega)]} \\
&\quad + \frac{1}{\epsilon^2} \|e_p\|_{L^2[0,T;L^2(\Omega)]} \|d_h\|_{L^2[0,T;L^2(\Omega)]} + \|d_p\|_{L^2[0,T;H^1(\Omega)]} \|e_h\|_{L^2[0,T;H^1(\Omega)]} \\
&\quad + \frac{C}{\epsilon^2} (\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]} + \|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}) \\
&\quad \quad \times \|d_h\|_{L^4[0,T;L^4(\Omega)]} (\|u_h\|_{L^4[0,T;L^4(\Omega)]} + \|u_p\|_{L^4[0,T;L^4(\Omega)]}) \\
&\leq \frac{C}{\epsilon^2} (\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \|e_p\|_{L^2[0,T;L^2(\Omega)]} \|d_h\|_{L^2[0,T;H^1(\Omega)]} \\
&\quad + \frac{1}{\epsilon^2} \|e_p\|_{L^2[0,T;L^2(\Omega)]} \|d_h\|_{L^2[0,T;L^2(\Omega)]} + \|d_p\|_{L^2[0,T;H^1(\Omega)]} \|e_h\|_{L^2[0,T;H^1(\Omega)]} \\
&\quad + \frac{C}{\epsilon^2} (\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]} + \|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}) \|d_h\|_{L^4[0,T;L^4(\Omega)]} \\
(5.20) \quad &+ \frac{1}{\epsilon^2} (\|d_h\|_{L^2[0,T;L^2(\Omega)]} + \|d_p\|_{L^2[0,T;L^2(\Omega)]}) \|e_h\|_{L^2[0,T;L^2(\Omega)]}.
\end{aligned}$$

where at the last step we have used the fact that $\|u_h\|_{L^4[0,T;L^4(\Omega)]}$ (due to Lemma 3.2) and $\|u_p\|_{L^4[0,T;L^4(\Omega)]}$ are bounded independent of ϵ . For the later, we note that for τ, h satisfying $\tau^{1/2} + h \leq \frac{1}{(\|u\|_{L^2[0,T;H^2(\Omega)]} + \|u_t\|_{L^2[0,T;L^2(\Omega)]})^2}$, using (2.3), (5.5), and (5.4) we easily deduce,

$$\begin{aligned}
\|u_p\|_{L^4[0,T;L^4(\Omega)]} &\leq \|u_p - u\|_{L^4[0,T;L^4(\Omega)]} + \|u\|_{L^4[0,T;L^4(\Omega)]} \\
&\leq \|u_p - u\|_{L^\infty[0,T;H^1(\Omega)]}^{1/2} \|u_p - u\|_{L^2[0,T;H^1(\Omega)]}^{1/2} + C \\
(5.21) \quad &\leq C(\tau^{1/2} + h)^{1/2} (\|u\|_{L^2[0,T;H^2(\Omega)]} + \|u_t\|_{L^2[0,T;L^2(\Omega)]}) + C \leq C,
\end{aligned}$$

where C denotes a constant independent of τ, h, ϵ . Lemma 5.4, implies that for $\tau^{1/2} + h \leq \epsilon$

$$\begin{aligned}
\|d_h\|_{L^2[0,T;H^1(\Omega)]} &\leq C \left(\|d_p\|_{L^2[0,T;H^1(\Omega)]} + \frac{1}{\epsilon} \|d_p\|_{L^2[0,T;L^2(\Omega)]} \right) \\
&\leq C \left((\tau^{1/2} + h) + \frac{1}{\epsilon} (\tau + h^2) \right) (\|\psi\|_{L^2[0,T;H^2(\Omega)]} + \|\psi_t\|_{L^2[0,T;L^2(\Omega)]}) \\
&\leq C \left((\tau^{1/2} + h) + \frac{1}{\epsilon} (\tau + h^2) \right) \|e_h\|_{L^2[0,T;L^2(\Omega)]} \leq C(\tau^{1/2} + h) \|e_h\|_{L^2[0,T;L^2(\Omega)]}.
\end{aligned}$$

In addition, we also have that

$$\begin{aligned} \|d_h\|_{L^2[0,T;L^2(\Omega)]} &\leq C\left((\tau^{1/2} + h)\|d_p\|_{L^2[0,T;H^1(\Omega)]} + \|d_p\|_{L^2[0,T;L^2(\Omega)]}\right) \\ &\leq C\left((\tau^{1/2} + h)^2 + (\tau + h^2)\right)(\|\psi\|_{L^2[0,T;H^2(\Omega)]} + \|\psi_t\|_{L^2[0,T;L^2(\Omega)]}) \\ &\leq C\left((\tau^{1/2} + h)^2 + (\tau + h^2)\right) \leq C(\tau + h^2)\|e_h\|_{L^2[0,T;L^2(\Omega)]}. \end{aligned}$$

Now we turn our attention to the term $\|d_h\|_{L^4[0,T;L^4(\Omega)]}$. We distinguish two cases:

- (1) Let $d = 2$. Then, the Gagliardo-Nirenberg interpolation inequality implies that,

$$\begin{aligned} \|d_h\|_{L^4[0,T;L^4(\Omega)]}^2 &\leq C\left(\int_0^T \|d_h\|_{L^2(\Omega)}^2 \|d_h\|_{H^1(\Omega)}^2 dt\right)^{1/2} \\ &\leq C\|d_h\|_{L^\infty[0,T;L^2(\Omega)]}\|d_h\|_{L^\infty[0,T;H^1(\Omega)]} \\ &\leq C(\tau^{1/2} + h)^2\|e_h\|_{L^2[0,T;L^2(\Omega)]}^2 \leq C(\tau + h^2)\|e_h\|_{L^2[0,T;L^2(\Omega)]}^2. \end{aligned}$$

- (2) Let $d = 3$. Then, the 3d Gagliardo-Nirenberg interpolation inequality implies that,

$$\begin{aligned} \|d_h\|_{L^4[0,T;L^4(\Omega)]}^2 &\leq C\left(\int_0^T \|d_h\|_{L^2(\Omega)} \|d_h\|_{H^1(\Omega)}^3 dt\right)^{1/2} \\ &\leq C\|d_h\|_{L^\infty[0,T;H^1(\Omega)]}\|d_h\|_{L^2[0,T;L^2(\Omega)]}^{1/2}\|d_h\|_{L^2[0,T;H^1(\Omega)]}^{1/2} \\ &\leq C(\tau^{1/2} + h)^{1/2}(\tau + h^2)^{1/2}\|e_h\|_{L^2[0,T;L^2(\Omega)]}^2. \end{aligned}$$

We note also that for $\tau + h^2 \leq \frac{\epsilon^2}{4}$ we obtain

$$(\|d_h\|_{L^2[0,T;L^2(\Omega)]} + \|d_p\|_{L^2[0,T;L^2(\Omega)]})\|e_h\|_{L^2[0,T;L^2(\Omega)]} \leq \frac{1}{2}\|e_h\|_{L^2[0,T;L^2(\Omega)]}^2.$$

Therefore, substituting the above bounds into (5.20), we finally obtain the desired estimates using standard algebra upon noting that $\tau^{1/2} + h \leq \epsilon^2$: For instance, for $d = 3$, we observe that replacing all d_h terms, we infer:

$$\begin{aligned} &(1/2)\|e_h\|_{L^2[0,T;L^2(\Omega)]}^2 \\ &\leq \frac{C}{\epsilon^2}(\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2)\|e_p\|_{L^2[0,T;L^2(\Omega)]}(\tau^{1/2} + h)\|e_h\|_{L^2[0,T;L^2(\Omega)]} \\ &\quad + \frac{1}{\epsilon^2}\|e_p\|_{L^2[0,T;L^2(\Omega)]}(\tau + h^2)\|e_h\|_{L^2[0,T;L^2(\Omega)]} \\ &\quad + C(\tau^{1/2} + h)\|e_h\|_{L^2[0,T;H^1(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]} \\ &\quad + \frac{C}{\epsilon^2}(\tau^{1/2} + h)^{\frac{3}{4}}(\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]} + \|e_h u_p\|_{L^2[0,T;L^2(\Omega)]})\|e_h\|_{L^2[0,T;L^2(\Omega)]}. \end{aligned}$$

The estimate now follows by standard algebra. For $d = 2$ the estimate follows by replacing the appropriate bound of $\|d_h\|_{L^4[0,T;L^4(\Omega)]}$. \square

5.3. Best approximation error estimates. Now, we are ready to proceed with the main estimate, using a boot-strap argument.

Theorem 5.6. Suppose that $\|f\|_{L^2[0,T;L^2(\Omega)]}$, and $\|u_0\|_{H^1(\Omega)}$. Suppose also that for any choice of τ, h and for an algebraic constant C_0 , satisfy

- (1) $\tau^{1/2} + h \leq \min\{C_0\epsilon^2, \frac{1}{(\|u\|_{L^2[0,T;H^2(\Omega)]} + \|u_t\|_{L^2[0,T;L^2(\Omega)]})^2}\}$ when $d = 2$,

$$(2) \quad \tau^{1/2} + h \leq \min\{C_0\epsilon^{8/3}, \frac{1}{(\|u\|_{L^2[0,T;H^2(\Omega)]} + \|u_t\|_{L^2[0,T;L^2(\Omega)]})^2}\} \text{ when } d = 3.$$

Then, there exists a constant (still) denoted by C depending only upon Ω , and C_k but independent of ϵ , such that,

$$\begin{aligned} & \|e_h\|_{L^2[0,T;H^1(\Omega)]}^2 + (1/\epsilon^2)(\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]}^2 + \|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}^2) + \|e_{h-}^N\|_{L^2(\Omega)}^2 \\ & + (1/\epsilon^2)\|e_h\|_{L^4[0,T;L^4(\Omega)]}^4 + \sum_{i=1}^{N-1} \|e_h^i\|_{L^2(\Omega)}^2 \\ & \leq C(1/\epsilon^2)\left(\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2\right)\|e_p\|_{L^2[0,T;H^1(\Omega)]}^2 \\ & + \|e_p\|_{L^2[0,T;L^2(\Omega)]}^2. \end{aligned}$$

Suppose also that (2.2) holds when $k \geq 1$. Then, there exists a constant C depending only upon Ω , and C_k such that

$$\begin{aligned} \|e_h\|_{L^\infty[0,T;L^2(\Omega)]}^2 & \leq C(1/\epsilon^2)\left(\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2\right)\|e_p\|_{L^2[0,T;H^1(\Omega)]}^2 \\ & + \|e_p\|_{L^2[0,T;L^2(\Omega)]}^2. \end{aligned}$$

Proof. Step 1: Estimate at the energy norm: Since, we have already obtained a bound on $\|e_h\|_{L^2[0,T;L^2(\Omega)]}$ with constant depending polynomially upon $1/\epsilon$, we may return to the orthogonality condition (5.7) and set $w_h = e_h$. Then, for every $n = 1, \dots, N$, we have:

$$\begin{aligned} & \frac{1}{2}\|e_{h-}^n\|_{L^2(\Omega)}^2 + C \int_{t^{n-1}}^{t^n} \|e_h\|_{H^1(\Omega)}^2 dt + \frac{1}{2}\|e_h^{n-1}\|_{L^2(\Omega)}^2 \\ & + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} (\|e_h u_h\|_{L^2(\Omega)}^2 + \|e_h u_p\|_{L^2(\Omega)}^2) dt + \|e_h^{n-1}\|_{L^2(\Omega)}^2 \\ & \leq \frac{1}{2}\|e_{h-}^{n-1}\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_h\|_{L^2(\Omega)}^2 dt \\ (5.22) \quad & + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} (|(e_p(u_p^2 + u^2 + u_p u), e_h)| + |(e_p, e_h)|) dt. \end{aligned}$$

It remains to bound the last two terms: First, we note that Hölder's and Young's inequalities imply

$$\begin{aligned} & \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} |(e_p(u_p^2 + u^2 + u_p u), e_h)| dt \\ & \leq \frac{C}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_p\|_{L^6(\Omega)} (\|u_p\|_{L^6(\Omega)}^2 + \|u\|_{L^6(\Omega)}^2) \|e_h\|_{L^2(\Omega)} dt \\ & \leq \frac{C}{\epsilon^2} (\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \int_{t^{n-1}}^{t^n} \|e_p\|_{H^1(\Omega)}^2 dt + \frac{C}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_h\|_{L^2(\Omega)}^2 dt. \end{aligned}$$

Substituting the last inequality into (5.22) we obtain,

$$\begin{aligned}
& \frac{1}{2} \|e_{h-}^n\|_{L^2(\Omega)}^2 + \frac{C}{2} \int_{t^{n-1}}^{t^n} \|e_h\|_{H^1(\Omega)}^2 dt + \frac{1}{2} \|[e_h^{n-1}]\|_{L^2(\Omega)}^2 \\
& \quad + (1/2\epsilon^2) \int_{t^{n-1}}^{t^n} (\|e_h u_h\|_{L^2(\Omega)}^2 + \|e_h u_p\|_{L^2(\Omega)}^2) dt \\
& \leq \frac{1}{2} \|e_{h-}^{n-1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_h\|_{L^2(\Omega)}^2 dt + \frac{C}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_p\|_{L^2(\Omega)}^2 dt \\
& \quad + \frac{C}{\epsilon^2} (\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \int_{t^{n-1}}^{t^n} \|e_p\|_{H^1(\Omega)}^2 dt
\end{aligned}$$

It remains to replace the term $(1/\epsilon^2) \int_{t^{n-1}}^{t^n} \|e_h\|_{L^2(\Omega)}^2 dt$ by Proposition 5.5. We treat the case $d = 3$, while the case $d = 2$ can be treated similarly. First, note that the bound of Proposition 5.5, implies that:

$$\begin{aligned}
& \frac{1}{2} \|e_{h-}^n\|_{L^2(\Omega)}^2 + \frac{C}{2} \int_{t^{n-1}}^{t^n} \|e_h\|_{H^1(\Omega)}^2 dt + \|[e_h^{n-1}]\|_{L^2(\Omega)}^2 \\
& \quad + \frac{1}{2\epsilon^2} \int_{t^{n-1}}^{t^n} (\|e_h u_h\|_{L^2(\Omega)}^2 + \|e_h u_p\|_{L^2(\Omega)}^2) dt \\
& \leq \frac{1}{2} \|e_{h-}^{n-1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_p\|_{L^2(\Omega)}^2 dt \\
& \quad + \frac{C}{\epsilon^2} (\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \int_{t^{n-1}}^{t^n} \|e_p\|_{H^1(\Omega)}^2 dt \\
& \quad + \frac{1}{\epsilon^2} (\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \int_{t^{n-1}}^{t^n} \|e_p\|_{L^2(\Omega)}^2 dt \\
& \quad + \frac{C}{\epsilon^2} (\tau^{1/2} + h)^2 \int_{t^{n-1}}^{t^n} \|e_h\|_{H^1(\Omega)}^2 dt \\
& \quad + \frac{C}{\epsilon^6} (\tau^{1/2} + h)^{3/2} \int_{t^{n-1}}^{t^n} (\|e_h u_h\|_{L^2(\Omega)}^2 + \|e_h u_p\|_{L^2(\Omega)}^2) dt.
\end{aligned}$$

Now, noting that we may choose τ, h in order to hide the last two terms on the left.

In particular, if $\frac{(\tau^{1/2}+h)^2}{\epsilon^2} \leq \frac{C}{4}$, and $\frac{C}{\epsilon^6} (\tau^{1/2} + h)^{3/2} \leq \frac{1}{4\epsilon^2}$ finally obtain,

$$\begin{aligned}
& \frac{1}{2} \|e_{h-}^n\|_{L^2(\Omega)}^2 + \frac{C}{4} \int_{t^{n-1}}^{t^n} \|e_h\|_{H^1(\Omega)}^2 dt + \frac{1}{2} \|[e_h^{n-1}]\|_{L^2(\Omega)}^2 \\
& \quad + \frac{1}{4\epsilon^2} \int_{t^{n-1}}^{t^n} (\|e_h u_h\|_{L^2(\Omega)}^2 + \|e_h u_p\|_{L^2(\Omega)}^2) dt \\
& \leq \frac{1}{2} \|e_{h-}^{n-1}\|_{L^2(\Omega)}^2 + \frac{C}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_p\|_{L^2(\Omega)}^2 dt \\
& \quad + \frac{C}{\epsilon^2} (\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \int_{t^{n-1}}^{t^n} \|e_p\|_{H^1(\Omega)}^2 dt \\
& \quad + \frac{1}{\epsilon^2} (\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \int_{t^{n-1}}^{t^n} \|e_p\|_{L^2(\Omega)}^2 dt.
\end{aligned}$$

which implies the first estimate. It is clear that the bounds on $\|e_u u_h\|_{L^2[0,T;L^2(\Omega)]}$ and on $\|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}$ imply a similar estimate for $\|e_h\|_{L^4[0,T;L^4(\Omega)]}$, since

$$\begin{aligned} (1/\epsilon^2) \int_0^T \|e_h\|_{L^4(\Omega)}^4 dt &\leq (2/\epsilon^2) \int_0^T \int_{\Omega} |e_h|^2 (|u_h|^2 + |u_p|^2) dx dt \\ &\leq (2/\epsilon^2) \int_0^T (\|e_h u_h\|_{L^2(\Omega)}^2 + \|e_h u_p\|_{L^2(\Omega)}^2) dt. \end{aligned}$$

Step 2: Estimates at arbitrary time points: We proceed to the estimate at arbitrary time-points. We use similar ideas to the proof of Proposition 4.3. For fixed $t \in [t^{n-1}, t^n]$ and $z_h \in U_h$ we set $w_h(s) = z_h \rho(s)$ into (5.7), with $\rho(s) \in \mathcal{P}_k[t^{n-1}, t^n]$ such that

$$\rho(t^{n-1}) = 1, \quad \int_{t^{n-1}}^{t^n} \rho q = \int_{t^{n-1}}^t q, \quad q \in \mathcal{P}_{k-1}[t^{n-1}, t^n].$$

From Lemma 4.2 we deduce that $\|\rho\|_{L^\infty} \leq C_k$, with C_k independent of t , and

$$\begin{aligned} &\int_{t^{n-1}}^{t^n} \langle e_{ht}, w_h \rangle ds + (e_{h+}^{n-1} - e_{h-}^{n-1}, w_{h+}^{n-1}) \\ &= \int_{t^{n-1}}^t \langle e_{ht}, z_h \rangle ds + (e_{h+}^{n-1} - e_{h-}^{n-1}, \rho(t^{n-1}) z_h) = (e_h(t) - e_{h-}^{n-1}, z_h). \end{aligned}$$

Therefore, integrating by parts (in time), (5.7), setting $w_h(s) = z_h \rho(s)$, using the above equality and standard algebra, we obtain:

$$\begin{aligned} (e_h(t) - e_{h-}^{n-1}, z_h) &\leq C_k \left[\int_{t^{n-1}}^{t^n} \int_{\Omega} |\nabla e_h| |\nabla z_h| dx ds \right. \\ &\quad + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} \int_{\Omega} (|e_h|(|u_h|^2 + |u_p|^2)|z_h| + |e_h||z_h|) dx ds \\ &\quad \left. + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} \int_{\Omega} (|e_p|(|u_p|^2 + |u|^2)|z_h| + |e_p||z_h|) dx ds \right]. \end{aligned} \quad (5.23)$$

Adding and subtracting u_p, u , and using standard algebra, we may bound

$$\int_{t^{n-1}}^{t^n} \int_{\Omega} |e_h|(|u_h|^2 + |u_p|^2)|z_h| dx ds \leq C \int_{t^{n-1}}^{t^n} \int_{\Omega} (|e_h|^3 + |e_h||u_p - u|^2 + |e_h||u|^2)|z_h| dx ds.$$

Hence, using Hölder's inequality into (5.23) we derive

$$\begin{aligned} &\langle e_h(t) - e_{h-}^{n-1}, z_h \rangle \\ &\leq C_k \left[\int_{t^{n-1}}^{t^n} \|\nabla e_h\|_{L^2(\Omega)} \|\nabla z_h\|_{L^2(\Omega)} + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_h\|_{L^2(\Omega)} \|z_h\|_{L^2(\Omega)} ds \right. \\ &\quad + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} \left(\|e_h\|_{L^4(\Omega)}^3 \|z_h\|_{L^4(\Omega)} + \|e_h\|_{L^4(\Omega)} \|e_p^2\|_{L^2(\Omega)} \|z_h\|_{L^4(\Omega)} \right) dt \\ &\quad + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_h\|_{L^6(\Omega)} \|u^2\|_{L^3(\Omega)} \|z_h\|_{L^2(\Omega)} ds \\ &\quad + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_p\|_{L^6(\Omega)} \|u_p^2 + u^2 + u_p u\|_{L^3(\Omega)} \|z_h\|_{L^2(\Omega)} ds \\ &\quad \left. + \frac{1}{\epsilon^2} \int_{t^{n-1}}^{t^n} \|e_p\|_{L^2(\Omega)} \|z_h\|_{L^2(\Omega)} ds \right]. \end{aligned} \quad (5.24)$$

Noting that z_h is independent of t , and standard algebra implies that

$$\begin{aligned}
& \langle e_h(t) - e_{h-}^{n-1}, z_h \rangle \\
& \leq C_k \left[\|z_h\|_{H^1(\Omega)} \int_{t^{n-1}}^{t^n} \|\nabla e_h\|_{L^2(\Omega)} ds + \frac{1}{\epsilon^2} \|z_h\|_{L^2(\Omega)} \int_{t^{n-1}}^{t^n} \|e_h\|_{L^2(\Omega)} ds \right. \\
& \quad + \frac{1}{\epsilon^2} \|z_h\|_{L^4(\Omega)} \int_{t^{n-1}}^{t^n} \|e_h\|_{L^4(\Omega)}^3 ds + \frac{1}{\epsilon^2} \|z_h\|_{L^4(\Omega)} \int_{t^{n-1}}^{t^n} \|e_h\|_{L^4(\Omega)} \|e_p\|_{L^4(\Omega)}^2 ds \\
& \quad + \frac{1}{\epsilon^2} \|z_h\|_{L^2(\Omega)} \int_{t^{n-1}}^{t^n} \|e_h\|_{L^6(\Omega)} \|u\|_{L^6(\Omega)}^2 ds \\
& \quad + \frac{1}{\epsilon^2} \|z_h\|_{L^2(\Omega)} \int_{t^{n-1}}^{t^n} \|e_p\|_{L^6(\Omega)} (\|u_p\|_{L^6(\Omega)}^2 + \|u\|_{L^6(\Omega)}^2) ds \\
& \quad \left. + \frac{1}{\epsilon^2} \|z_h\|_{L^2(\Omega)} \int_{t^{n-1}}^{t^n} \|e_p\|_{L^2(\Omega)} ds \right]. \tag{5.25}
\end{aligned}$$

Using once more Hölder's inequality and the fact that $u, u_p \in L^\infty[0, T; H^1(\Omega)]$, we deduce with different constant C_k (independent of ϵ):

$$\begin{aligned}
& \langle e_h(t) - e_{h-}^{n-1}, z_h \rangle \leq C_k \left[\|z_h\|_{H^1(\Omega)} \tau_n^{1/2} \|\nabla e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \right. \\
& \quad + \frac{\tau_n^{1/2}}{\epsilon^2} \|z_h\|_{L^2(\Omega)} \|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} + \frac{\tau_n^{1/4}}{\epsilon^2} \|z_h\|_{L^4(\Omega)} \|e_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^3 \\
& \quad + \frac{\tau_n^{1/4}}{\epsilon^2} \|z_h\|_{L^4(\Omega)} \|e_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]} \|e_p\|_{L^4[0, T; L^4(\Omega)]}^2 \\
& \quad + \frac{\tau_n^{1/2}}{\epsilon^2} \|z_h\|_{L^2(\Omega)} \|e_h\|_{L^2[t^{n-1}, t^n; L^6(\Omega)]} \|u\|_{L^\infty[0, T; L^6(\Omega)]}^2 \\
& \quad + \frac{\tau_n^{1/2}}{\epsilon^2} \|z_h\|_{L^2(\Omega)} \|e_p\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} (\|u_p\|_{L^\infty[0, T; L^6(\Omega)]}^2 + \|u\|_{L^\infty[0, T; L^6(\Omega)]}^2) \\
& \quad \left. + \frac{\tau_n^{1/2}}{\epsilon^2} \|z_h\|_{L^2(\Omega)} \|e_p\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \right].
\end{aligned}$$

Setting $z_h = e_h(t)$ and integrating with respect to time, using Hölder's inequality to bound $\int_{t^{n-1}}^{t^n} \|e_h(t)\|_{L^4(\Omega)} dt \leq \tau^{3/4} \|e_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}$, and standard calculations, we derive,

$$\begin{aligned}
& (5.26) \\
& \int_{t^{n-1}}^{t^n} \|e_h(t)\|_{L^2(\Omega)}^2 dt \leq \|e_{h-}^{n-1}\|_{L^2(\Omega)} \tau_n^{1/2} \|e_h(t)\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \\
& \quad + C_k \left[\tau_n \|e_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]}^2 + \frac{\tau_n}{\epsilon^2} \|e_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^4 \right. \\
& \quad + \frac{\tau_n}{\epsilon^2} \|e_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^2 \|e_p\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^2 \\
& \quad + \frac{\tau_n}{\epsilon^2} \|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \|e_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} \|u\|_{L^\infty[0, T; L^6(\Omega)]}^2 \\
& \quad + \frac{\tau_n}{\epsilon^2} \|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \|e_p\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} (\|u_p\|_{L^\infty[0, T; L^6(\Omega)]}^2 + \|u\|_{L^\infty[0, T; L^6(\Omega)]}^2) \\
& \quad \left. + \frac{\tau_n}{\epsilon^2} \|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \|e_p\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \right].
\end{aligned}$$

For the first term of the left hand side, using Young's inequality, we obtain:

$$\begin{aligned} & \|e_{h-}^{n-1}\|_{L^2(\Omega)} \tau_n^{1/2} \|e_h(t)\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \\ & \leq \frac{1}{4} \|e_h(t)\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]}^2 + C \tau_n \|e_{h+}^{n-1}\|_{L^2(\Omega)}^2. \end{aligned}$$

For the fourth term, we note that using Young's inequality, we obtain

$$\begin{aligned} & \frac{\tau_n}{\epsilon^2} \|e_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^2 \|e_p\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^2 \\ & \leq \frac{\tau_n}{\epsilon^2} \|e_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^4 + \frac{\tau_n}{\epsilon^2} \|e_p\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^4. \end{aligned}$$

For the fifth term, we note that (2.2), implies that $\|u\|_{L^\infty[0, T; L^6(\Omega)]} \leq C$ (where C is independent of ϵ), and hence we obtain,

$$\begin{aligned} & \frac{\tau_n}{\epsilon^2} \|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \|e_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} \|u\|_{L^\infty[0, T; L^6(\Omega)]}^2 \\ & \leq \frac{\tau_n}{\epsilon^4} \|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]}^2 + \tau_n \|e_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]}^2. \end{aligned}$$

For the last two terms, using similar algebra, we deduce,

$$\begin{aligned} & \frac{\tau_n}{\epsilon^2} \|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \|e_p\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} (\|u_p\|_{L^\infty[0, T; L^6(\Omega)]}^2 + \|u\|_{L^\infty[0, T; L^6(\Omega)]}^2) \\ & \leq \frac{\tau_n}{\epsilon^4} \|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]}^2 \\ & \quad + \tau_n \|e_p\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]}^2 \times (\|u_p\|_{L^\infty[0, T; L^6(\Omega)]}^2 + \|u\|_{L^\infty[0, T; L^6(\Omega)]}^2), \\ & \frac{\tau_n}{\epsilon^2} \|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \|e_p\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \\ & \leq \frac{\tau_n}{\epsilon^4} \|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]}^2 + \tau_n \|e_p\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]}^2. \end{aligned}$$

Note that choosing $\frac{C_k \tau_n}{\epsilon^4} \leq \frac{1}{8}$, we may hide all $\|e_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]}^2$ of (5.26) on the left. Hence, dividing by τ_n the resulting inequality and using an inverse estimate in time, we arrive at,

$$\begin{aligned} \|e_h\|_{L^\infty[t^{n-1}, t^n; L^2(\Omega)]}^2 & \leq C_k \left(\|e_{h+}^{n-1}\|_{L^2(\Omega)}^2 + \|e_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]}^2 + \frac{1}{\epsilon^2} \|e_h\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^4 \right. \\ & \quad + \|e_p\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]}^2 (\|u_p\|_{L^\infty[0, T; L^6(\Omega)]}^2 + \|u\|_{L^\infty[0, T; L^6(\Omega)]}^2) + \|e_p\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]}^2 \\ & \quad \left. + \frac{1}{\epsilon^2} \|e_p\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^4 \right). \end{aligned}$$

Now, note that

$$\|e_p\|_{L^4[t^{n-1}, t^n; L^4(\Omega)]}^4 \leq (\|u_p\|_{L^\infty[t^{n-1}, t^n; L^4(\Omega)]}^2 + \|u\|_{L^\infty[t^{n-1}, t^n; H^1(\Omega)]}^2) \|e_p\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]}^2.$$

Hence, the desired estimate now follows by replacing the bounds of $\|e_{h+}^{n-1}\|_{L^2(\Omega)}^2$, $\frac{1}{\epsilon^2} \|e_h\|_{L^4[0, T; L^4(\Omega)]}^4$, $\|e_h\|_{L^2[0, T; H^1(\Omega)]}^2$. \square

Remark 5.7. Below we state few remarks regarding the previous estimates.

- (1) The estimate at arbitrary time points results to a best-approximation result by using triangle inequality. In addition, the dependence of the constant upon $\frac{1}{\epsilon}$ doesn't deteriorate further, despite the fact that we treat schemes of arbitrary order, provided that the natural assumption $\frac{1}{\epsilon^2} \|(u_0^2 - 1)^2\|_{L^1(\Omega)} \leq C$ holds.
- (2) Our estimate at the energy norm at partition points is valid even without assuming the bound $\frac{1}{\epsilon^2} \|(u_0^2 - 1)^2\|_{L^1(\Omega)} \leq C$ (with C independent of ϵ).

- (3) Due to stability estimate (5.5), the norm of $\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}$ is estimated by the norm of $\|u\|_{L^2[0,T;H^2(\Omega)]}$, $\|u_t\|_{L^2[0,T;L^2(\Omega)]}$ and $\|u\|_{L^\infty[0,T;H^1(\Omega)]}$, and hence it scales as $\frac{1}{\epsilon}$. Therefore, the assumption τ, h can be written as:
- (a) $\tau^{1/2} + h \leq C_0 \epsilon^2$ when $d = 2$,
 - (b) $\tau^{1/2} + h \leq C_0 \epsilon^{8/3}$ when $d = 3$.

The best approximation estimate now follows by triangle inequality.

Theorem 5.8. Suppose that (2.2) holds. Suppose also that τ, h satisfy $\tau^{1/2} + h \leq C\epsilon^2$ when $d = 2$, and $\tau^{1/2} + h \leq C\epsilon^{8/3}$ when $d = 3$ for some algebraic constant C . Then, there exists a constant \mathbf{C} depending only upon Ω , C_k and $\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2$ but independent of ϵ , and the such that,

$$\|e\|_{L^2[0,T;H^1(\Omega)]} + \|e\|_{L^\infty[0,T;L^2(\Omega)]} \leq \mathbf{C}(1/\epsilon) (\|e_p\|_{L^2[0,T;H^1(\Omega)]} + \|e_p\|_{L^\infty[0,T;L^2(\Omega)]}).$$

If in addition $u \in L^2[0, T; H^{l+1}(\Omega)]$, $u^{(k+1)} \in L^\infty[0, T; L^2(\Omega)]$ there exists a positive constant C that depends only upon Ω, C_k and it is independent of h, τ, ϵ , such that

$$\begin{aligned} & \|e\|_{L^2[0,T;H^1(\Omega)]} + \|e\|_{L^\infty[0,T;L^2(\Omega)]} \\ & \leq C(1/\epsilon^3) (h^l \|u\|_{L^2[0,T;H^{l+1}(\Omega)]} + \tau^{k+1} \|u^{(k+1)}\|_{L^\infty[0,T;L^2(\Omega)]}). \end{aligned}$$

Proof. Using triangle inequality we obtain the first estimate. Then, the rates of convergence follow by the estimates on e_p in $L^2[0, T; H^1(\Omega)]$, and $L^\infty[0, T; L^2(\Omega)]$ norms using Lemma 5.2 and 5.4. since $\|u_p\|_{L^\infty[0,T;H^1(\Omega)]} + \|u\|_{L^\infty[0,T;H^1(\Omega)]} \leq C/\epsilon$ by (5.5). \square

Proposition 5.9. Suppose that $k = 0, l = 1$. Let also τ, h satisfy $\tau^{1/2} + h \leq C\epsilon^2$ when $d = 2$, and $\tau^{1/2} + h \leq C\epsilon^{8/3}$ when $d = 3$ for some algebraic constant C . Then there exists a positive constant \mathbf{C} depending only upon Ω , C_k and $\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2$ but independent of h, τ, ϵ such that,

- (1) $\|e\|_{L^2[0,T;H^1(\Omega)]} + \|e\|_{L^\infty[0,T;L^2(\Omega)]} \leq \mathbf{C}(1/\epsilon^2)(\tau^{1/2} + h)$,
when $u \in L^2[0, T; H^2(\Omega)] \cap H^1[0, T; L^2(\Omega)]$,
- (2) $\|e\|_{L^2[0,T;H^1(\Omega)]} + \|e\|_{L^\infty[0,T;L^2(\Omega)]} \leq \mathbf{C}(1/\epsilon^2)(\tau + h)$,
when $u \in L^2[0, T; H^2(\Omega)] \cap H^1[0, T; H^1(\Omega)]$.

Proof. The estimates concerning the lowest order scheme follow directly from Theorem 5.6, and the approximation properties of e_p in $L^2[0, T; H^1(\Omega)]$ and $L^2[0, T; L^2(\Omega)]$ norms. Note that $\|u_p\|_{L^\infty[0,T;H^1(\Omega)]} \leq C/\epsilon$ by (5.5). \square

We conclude our estimates by presenting an improved estimate in $L^2[0, T; L^2(\Omega)]$ norm.

Theorem 5.10. Suppose that the assumptions of Theorem 5.8 hold. Let $d = 2$. Then, there exists a constant C independent of h, τ, ϵ such that,

$$\begin{aligned} \|e_h\|_{L^2[0,T;L^2(\Omega)]} & \leq C \left((\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) + \frac{1}{\epsilon^2} \right) \|e_p\|_{L^2[0,T;L^2(\Omega)]} \\ & \quad + \frac{1}{\epsilon^2} (\tau^{1/2} + h) (\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \|e_p\|_{L^2[0,T;H^1(\Omega)]}. \end{aligned}$$

Let $d = 3$. Then, there exists a constant C independent of h, τ, ϵ such that,

$$\begin{aligned} \|e_h\|_{L^2[0,T;L^2(\Omega)]} &\leq C \left((\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \frac{1}{\epsilon^2}) \|e_p\|_{L^2[0,T;L^2(\Omega)]} \right. \\ &\quad \left. + \frac{1}{\epsilon^2} (\tau^{1/2} + h)^{3/4} (\|u_p\|_{L^\infty[0,T;L^6(\Omega)]}^2 + \|u\|_{L^\infty[0,T;L^6(\Omega)]}^2) \|e_p\|_{L^2[0,T;H^1(\Omega)]} \right). \end{aligned}$$

Proof. Returning back to the estimate of Proposition 5.5, and replacing the norms of $\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]}$, $\|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}$ and $\|e_h\|_{L^2[0,T;H^1(\Omega)]}$ by the estimates of Theorem 5.8, we obtain the desired results using standard algebra. \square

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